Asymptotic Risk Factor Model with Volatility Factors

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1. INTRODUCTION

Introduction

- Three categories of models :
  1. The portfolio credit Value-at-Risk (VaR) models.
  2. The reduced form models.
  3. The structural models.
- The Asymptotic Single Risk Factor Model is the basis for :
  - ⇒ The analysis of credit risk.
  - ⇒ Performing the stress tests (including the possibility to account for a rating scale).

Motivation

⇒ The standard ASRF Model

$$V_{k,i,t} = \alpha_k - \sqrt{\rho} Z_t^* + \sqrt{1-\rho} u_{k,i,t}, k = 1, 2,$$

where $EZ_t^* = 0, VZ_t^* = 1$, $u's$ are i.i.d. standard normal, and ρ is interpreted as a correlation.

- A single factor model with linear effect of the factor.

- This basic model has been extended in the literature to include more than a single linear factor [see e.g. Gagliardini, Gourieroux (2005)].
  ⇒ But this extension does not include nonlinear factor such as volatility

⇒ The idea behind this paper is that nonlinear effects can be appropriately captured by introducing time varying volatility factors

Objective of the paper

In this paper we ...

- Extend the standard Asymptotic Single Risk Factor Model to a Risk Factor Model with both drift and volatility factors.
- Provide a calibration step based on the asymptotic principle (granularity theory) to derive consistent estimators of the parameters and consistent smoothed factor values.
- Discuss the consequences of using misspecified ASRF model, i.e. of neglecting volatility factors.
- Provide an application of the standard ASRF model and the Risk Factor Model with both drift and volatility factors in a stress testing framework for corporate credit portfolio by using comprehensive database managed within Credit Agricole Group.

2. THE RISK FACTOR MODEL

The Model

- Model defined by means of latent variables interpretable as log asset/liability ratios : Y^*.
- The distribution of these latent variables depends on the class of risk at the previous period.

  $k = 1$: investment grade $\rightarrow Y^*_{1,i,t}$
  $k = 2$: speculative grade $\rightarrow Y^*_{2,j,t}$
  $k = 3$: default (absorbing state)

- We assume :

$$\begin{cases} Y^*_{1,i,t} = \alpha_1 - \beta_1 Z_t + \gamma_{1t} u_{1,i,t}, i = 1, \dots, n_{1,t-1}, \\ Y^*_{2,j,t} = \alpha_2 - \beta_2 Z_t + \gamma_{2t} u_{2,j,t}, j = 1, \dots, n_{2,t-1}, \end{cases}$$

  where $u_{1,i,t}, u_{2,j,t}$ are i.i.d. standard normal.

- 3 types of factors :

  i) Z_t, linear factor
  ii) γ_{1t}, γ_{2t} nonlinear stochastic volatility factors.

The Model

- The conditional distribution of $Y^*_{k,i,t}$ is :

$$Y^*_{k,i,t}|Z_t, \gamma_{1t}, \gamma_{2t} \sim N(\alpha_k - \beta_k Z_t, \gamma^2_{kt}), k = 1, 2. \quad (1)$$

- The log asset/liability ratios are not directly observed.
- The observed variables are the new ratings :

  $Y_{k,i,t} = 1$, investment grade if $Y^*_{k,i,t} > c_1$,

  $Y_{k,i,t} = 2$, speculative grade if $c_1 > Y^*_{k,i,t} > c_2$,

  $Y_{k,i,t} = 3$, default if $c_2 > Y^*_{k,i,t}$,

  where c_1, c_2 are unknown thresholds.

The Model

Conditional Migration Probabilities

- From rating k to default :

$$\Pi_{k,3,t} = P(Y^*_{k,i,t} < c_2) = \Phi\left(\frac{c_2 - \alpha_k + \beta_k Z_t}{\gamma_{kt}}\right), k = 1, 2;$$

- From rating k to investment grade :

$$\Pi_{k,1,t} = P(Y^*_{k,i,t} > c_1) = 1 - \Phi\left(\frac{c_1 - \alpha_k + \beta_k Z_t}{\gamma_{kt}}\right), k = 1, 2;$$

- From rating k to speculative grade :

$$\Pi_{k,2,t} = 1 - \Pi_{k,1,t} - \Pi_{k,3,t}.$$

The Model

Conditional Migration Probabilities

These formulas can be used to see how the transformed probit migration probabilities depend on the factors :

$$\begin{cases} \Phi^{-1}(\Pi_{k,3,t}) & = & \dfrac{c_2 - \alpha_k + \beta_k Z_t}{\gamma_{kt}}, k = 1, 2, t = 1, \dots, T, \\ \Phi^{-1}(1 - \Pi_{k,1,t}) & = & \dfrac{c_1 - \alpha_k + \beta_k Z_t}{\gamma_{kt}}, k = 1, 2, t = 1, \dots, T. \end{cases}$$

The Model

Identification Restrictions

By appropriate drift and scaling of factors and parameters, we assume without loss of generality:

$\beta_1 = 1$, $c_2 = \alpha_1$, $\alpha_1 - \alpha_2 = 1$, and $c = c_1$ - $c_2 > 0$, $\beta = \beta_2$

$$\Rightarrow \Phi^{-1}(\Pi_{1,3,t}) = \frac{Z_t}{\gamma_{1t}},$$

$$\Rightarrow \Phi^{-1}(1 - \Pi_{1,1,t}) = \frac{c + Z_t}{\gamma_{1t}}.$$

$$\Rightarrow \Phi^{-1}(\Pi_{2,3,t}) = \frac{1 + \beta Z_t}{\gamma_{2t}},$$

$$\Rightarrow \Phi^{-1}(1 - \Pi_{2,1,t}) = \frac{c + 1 + \beta Z_t}{\gamma_{2t}}.$$

The Model

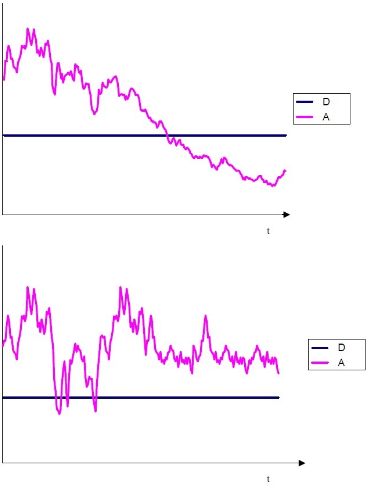
- The expressions of migration probabilities extend the standard ASRF formulas based on the assumption : $\gamma_{1t} = \gamma_{2t} = \gamma$, and the latent variables are defined with another identification restriction : $EZ_t^* = 0, VZ_t^* = 1$.

$$\rightarrow \ \Pi_{k,3,t} = \Phi\left(\frac{c_2 - \alpha_k}{\sqrt{1-\rho}} + \frac{\sqrt{\rho}}{\sqrt{1-\rho}} Z_t^*\right)$$

$$\rightarrow \ \Pi_{k,1,t} = 1 - \Phi\left(\frac{c_1 - \alpha_k}{\sqrt{1-\rho}} + \frac{\sqrt{\rho}}{\sqrt{1-\rho}} Z_t^*\right),$$

The Model

- The new model increases the number of factors and allows for different reasons of entering into the default state.



3. CALIBRATION

Calibration of RFM

Asymptotic Principle

- When the numbers of firms $n_{k,t}$ are large $\forall k, t$:
$$\hat{\Pi}_{k,l,t} \sim \Pi_{k,l,t}.$$
- Then for given thresholds c, β, we can use the closed form expressions :

$$\begin{aligned}
\frac{1}{\gamma_{1t}} &= \frac{\Phi^{-1}(1-\Pi_{1,1,t}) - \Phi^{-1}(\Pi_{1,3,t})}{c} \geq 0, \\
\frac{1}{\gamma_{2t}} &= \frac{\Phi^{-1}(1-\Pi_{2,1,t}) - \Phi^{-1}(\Pi_{2,3,t})}{c} \geq 0, \\
Z_t &= \frac{c\Phi^{-1}(\Pi_{1,3,t})}{\Phi^{-1}(1-\Pi_{1,1,t}) - \Phi^{-1}(\Pi_{1,3,t})} (\equiv Z_{1t}(c,\beta)) \\
&= 1/\beta[\frac{c\Phi^{-1}(\Pi_{2,3,t})}{\Phi^{-1}(1-\Pi_{2,1,t}) - \Phi^{-1}(\Pi_{2,3,t})} - 1](\equiv Z_{2t}(c,\beta)).
\end{aligned}$$

Calibration of RFM

The calibration is as follows :

- Plug in for $\hat{\Pi} \rightarrow \hat{Z}_{1t}(c, \beta), \hat{Z}_{2t}(c, \beta)$
- Estimation of the thresholds :

$$(\hat{c}, \hat{\beta}) = \arg\min_{c>0} \sum_{t=1}^{T} \left[\hat{Z}_{1t}(c, \beta) - \hat{Z}_{2t}(c, \beta)\right]^2 .$$

- Smoothed factor values:

$$\hat{Z}_t = \frac{1}{2} \left[\hat{Z}_{1t}(\hat{c}, \hat{\beta}) + \hat{Z}_{2t}(\hat{c}, \hat{\beta})\right] ;$$

- Then plug in for $\hat{\gamma}_{1t}, \hat{\gamma}_{2t}$.

Calibration of Standard RFM

The closed form expressions in the standard RFM :

$$\begin{cases} \Pi_{k,3,t} & = & \Phi\left(\dfrac{\Phi^{-1}(\Pi_{k,3})}{\sqrt{1-\rho}} + \dfrac{\sqrt{\rho}}{\sqrt{1-\rho}} Z_t^*\right), \\ \\ 1-\Pi_{k,1,t} & = & \Phi\left(\dfrac{\Phi^{-1}(1-\Pi_{k,1})}{\sqrt{1-\rho}} + \dfrac{\sqrt{\rho}}{\sqrt{1-\rho}} Z_t^*\right). \end{cases} \quad (2)$$

$$\begin{cases} \hat{Z}^*_{k,3,t}(\rho) & = & \dfrac{\sqrt{1-\rho}}{\sqrt{\rho}}\Phi^{-1}(\hat{\Pi}_{k,3,t}) - \dfrac{\Phi^{-1}(\hat{\Pi}_{k,3})}{\sqrt{\rho}}, \\ \\ \hat{Z}^*_{k,1,t}(\rho) & = & \dfrac{\sqrt{1-\rho}}{\sqrt{\rho}}\Phi^{-1}(1-\hat{\Pi}_{k,1,t}) - \dfrac{\Phi^{-1}(1-\hat{\Pi}_{k,1})}{\sqrt{\rho}}. \end{cases} \quad (3)$$

$k = 1, 2$

Calibration of Standard RFM

The calibration steps :

step 1 : : Estimate ρ

$$\hat{\rho} = \arg\min_{\rho} \sum_{t=1}^{T} \sum_{k=1}^{2} \sum_{l=1,3} \left[\hat{Z}^*_{k,l,t}(\rho) - \frac{1}{4} \sum_{k=1}^{2} \sum_{l=1,3} \hat{Z}^*_{k,l,t}(\rho) \right]^2 .$$

step 2 : The smoothed factor value is deduced by :

$$\hat{Z}^*_t = \frac{1}{4} \sum_{k=1}^{2} \sum_{l=1,3} \hat{Z}^*_{k,l,t}(\hat{\rho}).$$

Misspecified ASRFM

What are the consequences of implementing an ASRF model, when the true model is a RFM with volatility factors ?

- Intuitively :

$$\hat{Z}_t^* \simeq a_0 + a_1 \frac{1}{\gamma_{1t}} + a_2 \frac{1}{\gamma_{2t}} + a_3 \frac{Z_t}{\gamma_{1t}} + a_4 \frac{Z_t}{\gamma_{2t}}.$$

A pseudo factor linear combination of $\frac{1}{\gamma_{1t}}, \frac{1}{\gamma_{2t}}, \frac{Z_t}{\gamma_{1t}}, \frac{Z_t}{\gamma_{2t}}$.

4. ILLUSTRATION

The Data Set

- Corporate loans granted by the Credit Agricole Group.
- From December 2007 to December 2016.
- The internal ratings have been aggregated to get the 3 rating classes.
- Quarterly observations providing 35 quarterly migration matrices.

A Complete Migration Matrix

	A+	A	B+	B	C+	C	C-	D+	D	D-	E+	E	E-	F_Z
A+	89,39%	3,03%	1,52%	0%	6,06%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
A	0,63%	85,00%	3,75%	3,13%	4,38%	0,63%	0,00%	1,25%	0,63%	0,63%	0,00%	0,00%	0,00%	0,00%
B+	0,26%	0,26%	89,38%	3,28%	2,36%	2,10%	0,52%	0,66%	0,52%	0,26%	0,13%	0,00%	0,26%	0,00%
B	0,00%	0,50%	1,74%	84,67%	4,92%	3,30%	1,43%	1,43%	0,75%	0,44%	0,25%	0,06%	0,12%	0,37%
C+	0,06%	0,06%	0,56%	0,92%	89,40%	2,41%	2,72%	1,36%	1,00%	0,69%	0,22%	0,11%	0,11%	0,39%
C	0,02%	0,02%	0,08%	0,59%	2,03%	87,50%	4,13%	2,47%	1,56%	0,75%	0,34%	0,16%	0,15%	0,20%
C-	0,00%	0,02%	0,13%	0,06%	0,46%	1,91%	89,04%	3,72%	2,15%	1,10%	0,61%	0,20%	0,23%	0,38%
D+	0,00%	0,02%	0,04%	0,13%	0,40%	0,91%	2,92%	88,29%	3,93%	1,84%	0,71%	0,19%	0,23%	0,38%
D	0,01%	0,01%	0,02%	0,10%	0,24%	0,46%	1,58%	3,52%	87,19%	4,16%	1,14%	0,62%	0,49%	0,47%
D-	0,00%	0,01%	0,03%	0,02%	0,06%	0,26%	0,80%	1,79%	4,14%	87,74%	2,53%	0,95%	0,71%	0,95%
E+	0,00%	0,00%	0,02%	0,09%	0,11%	0,36%	0,64%	1,02%	2,51%	4,38%	85,26%	2,01%	1,68%	1,92%
E	0,00%	0,00%	0,00%	0,00%	0,06%	0,21%	0,30%	0,61%	1,73%	3,22%	3,58%	85,18%	2,98%	2,13%
E-	0,00%	0,05%	0,00%	0,05%	0,16%	0,16%	0,24%	0,79%	1,38%	1,57%	2,39%	2,28%	87,19%	3,72%
F_Z	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	100%

Figure: PIT Migration Matrix

	A+	A	B+	B	C+	C	C-	D+	D	D-	E+	E	E-	F_Z
A+	65,31%	6,46%	8,84%	5,44%	5,10%	1,70%	3,40%	1,70%	0,68%	1,02%	0,34%	0,00%	0,00%	0,00%
A	1,31%	53,28%	16,31%	9,74%	6,84%	4,98%	2,07%	2,21%	1,38%	0,83%	0,07%	0,41%	0,48%	0,07%
B+	0,50%	3,92%	47,60%	16,72%	10,12%	7,96%	4,49%	3,49%	2,21%	1,62%	0,61%	0,34%	0,34%	0,07%
B	0,11%	0,94%	6,96%	42,55%	14,87%	14,01%	8,34%	5,32%	3,27%	2,02%	0,75%	0,29%	0,36%	0,21%
C+	0,05%	0,21%	1,90%	6,69%	45,64%	16,49%	11,51%	6,86%	5,30%	3,01%	1,05%	0,39%	0,54%	0,39%
C	0,02%	0,07%	0,64%	2,24%	8,35%	44,15%	18,77%	11,55%	7,34%	4,09%	1,28%	0,51%	0,63%	0,35%
C-	0,01%	0,05%	0,26%	0,66%	3,02%	11,21%	44,84%	17,36%	11,57%	6,86%	2,13%	0,83%	0,72%	0,49%
D+	0,01%	0,03%	0,15%	0,41%	1,29%	4,64%	13,43%	43,66%	19,32%	10,93%	3,40%	1,13%	0,95%	0,64%
D	0,02%	0,03%	0,06%	0,20%	0,74%	2,14%	6,60%	15,15%	43,81%	20,47%	6,23%	2,14%	1,49%	0,93%
D-	0,00%	0,03%	0,04%	0,10%	0,36%	0,92%	2,83%	6,56%	17,31%	49,64%	13,22%	4,64%	2,51%	1,86%
E+	0,01%	0,02%	0,03%	0,07%	0,24%	0,51%	1,61%	3,11%	8,17%	24,02%	41,88%	11,16%	5,51%	3,66%
E	0,01%	0,02%	0,04%	0,10%	0,15%	0,42%	1,22%	2,53%	5,10%	14,92%	20,51%	37,41%	11,28%	6,30%
E-	0,02%	0,04%	0,08%	0,12%	0,28%	0,66%	1,47%	2,40%	3,65%	7,71%	9,62%	11,45%	52,24%	10,27%
F_Z	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	100,00%

Figure: TTC Migration Matrix

The Simplified Migration Matrix

	IG	SG	D
IG	93,79%	5,90%	0,31%
SG	2,21%	96,73%	1,06%
D	0,00%	0,00%	100%

PIT Migration Matrix

	IG	SG	D
IG	70,91%	28,69%	0,39%
SG	8,23%	89,26%	2,51%
D	0,00%	0,00%	100%

TTC Migration Matrix

Estimation Results

Two models are estimated :

1. The standard Single Risk Factor Model
   - Estimated ρ : $\hat{\rho} = 0.176$
   - ρ estimated by Basel formula : $\hat{\rho} = 0.165$
2. The model with volatility factors
   - Estimated c : $\hat{c} = 0.11$
   - Estimated β : $\hat{\beta} = 7.5$

Summary statistics for the factors

	Z_t^*	Z_t	$\hat{\gamma_{1t}}$	$\hat{\gamma_{2t}}$
Mean	-0.5021	-0.1187	0.0623	0.0248
STD	0.1431	0.1085	0.0097	0.0008

Table: Descriptive Statistics

For each volatility factor γ_{1t}, γ_{2t} is computed the standard deviation/mean ratio to check the hypothesis of constant volatility.

- For γ_{1t} : 0.15
- For γ_{2t} : 0.03

Both are significantly different from constant.

Summary statistics for the factors

It is also interesting to study the (unconditional) link between the 3 factors: $\hat{Z}_t$, $\hat{\gamma}_{1t}$, $\hat{\gamma}_{2t}$.

	$\hat{Z}_t$	$\hat{\gamma}_{1t}$	$\hat{\gamma}_{2t}$
$\hat{Z}_t$	1	-0.9367	0.2214
$\hat{\gamma}_{1t}$		1	0.0432
$\hat{\gamma}_{2t}$			1

Table: Correlation Matrix

Bivariate Plots

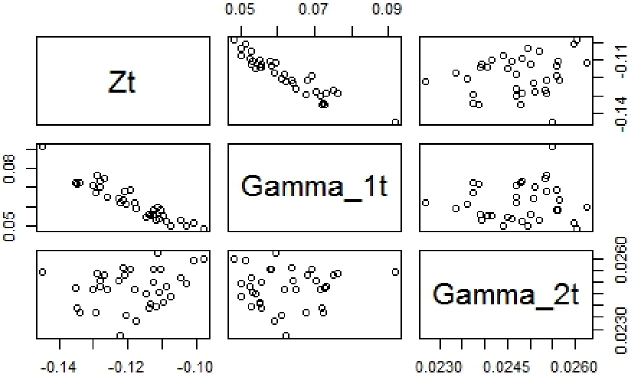


Figure: Bi-variate plots

Graphics

Graphs of factors (Annualized)

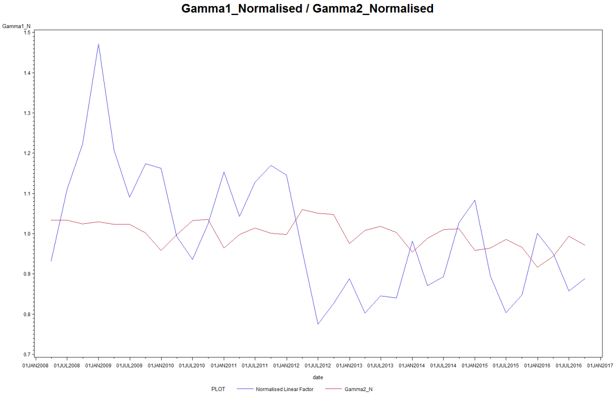


Figure: Evolution of the volatility factors

Graphics

Graphs of factors (Annualized)

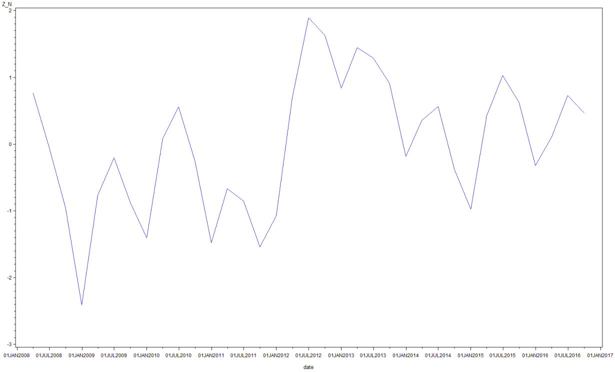


Figure: Evolution of the linear factor

Discussion on Misspecification

How to interpret the factor Z_t^* in the single risk factor model ?

Regress $\hat{Z}_t^*$ on $\frac{1}{\hat{\gamma_1}_t}, \frac{1}{\hat{\gamma_2}_t}, \frac{\hat{Z}_t}{\hat{\gamma_1}_t}, \frac{\hat{Z}_t}{\hat{\gamma_2}_t}$.

$$\begin{cases} \hat{Z}_t^* = 1.74^{***} + 0.16^{***}\frac{1}{\hat{\gamma_1}_t} - 0.02^{*}\frac{1}{\hat{\gamma_2}_t} + 2.67^{***}\frac{\hat{Z}_t}{\hat{\gamma_1}_t} - 0.24^{*}\frac{\hat{Z}_t}{\hat{\gamma_2}_t}, \\ R^2 adjusted = 0.87 \end{cases} \tag{4}$$

5. STRESS TESTS

Stress Testing Approach

As usual the stress tests are performed along the following steps :

i) Select a set of macro-variables to be stressed and define the directions and magnitude of the stresses (i.e. of the shocks).

ii) Estimate a dynamic model to relate the evolution of the factors to the macro-variables : Z_t^* for the Single Risk Factor model, Z_t, γ_{1t}, γ_{2t} for the model with volatility factors.

iii) Then apply the shocks to the macro-variables and look at their impact on first the factor, second the migration matrices.

iv) Use these stressed matrices to deduce the term structure of ratings.

The macro-variables

We consider a list of macro-variables associated with the scenarios proposed by the EBA. This macro-variables are :

- The French GDP growth rate
- The French inflation rate
- The change in French unemployment rate
- A long run interest rate (the 10 year OAT rate)
- The market index return (computed from the CAC40 index).

The change in real estate index is in the EBA scenarios, but is not introduced, since it is more relevant for mortgages than for corporate loans.

Link between the macro-variables and the factors : Dynamic Model

The linear factor

$$\begin{cases} \hat{Z}^*_t = -0.13^{***} + 0.73^{***}\,\hat{Z}^*{}_{t-1} - 5.65^{**}\,GDPg_t \\ \\ R^2\,adjusted = 0.68 \end{cases} \tag{5}$$

$$\begin{cases} \hat{Z}_t = -0.05^{**} + 0.61^{***}\,\hat{Z}_{t-1} + 0.69^{**}\,INFL_t - 0.28^{***}\,Euribor_t, \\ \\ R^2\,adjusted = 0.56 \end{cases} \tag{6}$$

Link between the macro-variables and the factors : Dynamic Model

The volatility factors

$$\begin{cases} \log \hat{\gamma_{1,t}} = -1.27^{***} + 0.56^{***} \log \hat{\gamma}_{1,t-1} - 6.93^{*} INFL_t + 5.03^{***} Euribor_t \\ R^2 adjusted = 0.70 \end{cases} \quad (7)$$

$$\begin{cases} \log \hat{\gamma_{2,t}} = -2.33^{***} + 0.37^{**} \log \hat{\gamma}_{2,t-1} + 0.72^{*} Euribor_t \\ R^2 adjusted = 0.32 \end{cases} \quad (8)$$

Scenarios and Stress Tests

- We consider the scenarios proposed in EBA(2016).

Macro-variables	Baseline (%)		Adverse (%)	
	2017	2018	2017	2018
OAT 10 year	1,3	1,4	2	2
GDP Growth	1,7	1,6	-1,1	0,6
Inflation	1,3	1,6	0,5	1
Unemployment rate	10,2	10,1	10,6	11,1
Residential property prices	1,5	2,3	-4,3	-1,5
Stock price shocks	-	-		

EBA Stress Test Scenarios

Scenarios and Stress Tests

- **For each scenario we deduce :**
  - The stressed factor values of Z* in the ASRF model and of Z and volatility factors in the second RFM for the futures years, from the complete regression models.
  - The stressed future migration matrices or ratings.

6. CONCLUSION

Conclusion

- We propose a RFM with both drift and volatility factors :
  - Extend the standard ASRF
  - Introduce volatility factors i.e. nonlinear factors.
  - Provide an estimation and calibration method which is valid for large cross-sections and even for small number of observation dates.
- We illustrate the models [RFM with volatility factors and Standard ASRF Model] in a stress testing exercise for a corporate credit portfolio.

THANK YOU