Composite Indirect Inference with Applications to Corporate Risks

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1. INTRODUCTION

The parametric models for credit risk are often difficult to estimate by maximum likelihood, say, due to :

- the large number of data (big data)
- the large number of parameters (high dimensionality)
- the complicated nonlinear dynamics
- the unobservable factors (systematic and/or specific)

This implies complicated log-likelihood functions with huge dimensional integrals difficult to compute numerically.

The aim of this paper is to explain :

- how to combine misspecified models with a small dimension (instrumental models) and their associated misspecified log-likelihoods
  and
- simulation approaches to adjust for the misspecification bias.

Thus we make a bridge between

• the literature on misspecified log-likelihood, especially the composite likelihood approach.

Cox, Reid (2004) :"A Note on Pseudo Likelihood Constructed From Marginal Densities", Biometrika, 91, 729-737.
Varin, Reid, Firth (2011) : "An Overview of Composite Likelihood Methods", Statistica Sinica, 21, 5-42.

• the literature on indirect inference :

Gourieroux, Monfort, Renault (1993) : "Indirect Inference", Journal of Applied Econometrics, 8, 85-118.
Gourieroux, Monfort (1996) : "Simulation Based Econometric Methods", Oxford University Press.

For expository purpose the presentation is for credit risk analysis.

2. STANDARD ANALYSIS OF RATING MIGRATION

2.1 The Model

• Observations of individual rating histories :

$$y_t = 1, \text{ investment grade,}$$
$$y_t = 0, \text{ speculative grade,;}$$

defined from a quantitative latent variable y_t^*, interpreted as a log asset/liability ratio [Merton (1974), Journal of Finance], by

$$y_t = 1, \text{if } y_t^* > c,$$
$$y_t = 0, \text{if } y_t^* < c.$$

• Latent dynamics

The model is completed by introducing the dynamics of the log asset/liability ratio :

$$y_t^* = m + \rho(y_{t-1}^* - m) + \sigma u_t,$$

where (u_t) is a sequence of iid $N(0, 1)$.

(includes Black, Scholes when $\rho = 1$)

• Identification restriction due to partial observation :

$$c = 0, \sigma = 1.$$

2.2 Marginal and Pairwise Distributions

The joint distribution of a rating history $y_t, t = 1, \ldots, T$ has a complicated form because of the autoregressive dynamics. But the marginal and pairwise distributions are under closed form.

- Marginal distribution of y_t

$$P[y_t = 1] = P[y_t > 0] = \Phi(m\sqrt{1-\rho^2}) \equiv \Phi(\delta),$$

where Φ is the cdf of the standard normal.

- Pairwise distribution of y_t, y_{t-h}

$$\begin{aligned} P[y_t = 1, y_{t-h} = 1] &= P[y_t^* > 0, y_{t-h}^* > 0] \\ &= \psi(m\sqrt{1-\rho^2}, m\sqrt{1-\rho^2}, \rho^h) \\ &= \psi(\delta, \delta, \rho^h), \end{aligned}$$

where ψ is the joint cdf of $N\left[0, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right]$.

2.3 The Standard Estimation Method

Calibrate δ, ρ, that are m, ρ by solving

$$\frac{1}{T}\sum_{t=1}^{T} y_t = \Phi(\delta), \quad (1)$$

$$\frac{1}{T}\sum_{t=1}^{T} y_t y_{t-1} = \psi(\delta, \delta, \rho), \quad (2)$$

recursively.

First, find $\hat{\delta}$ from (1),
Second, plug in $\hat{\delta}$ in (2) and solve (2) with respect to ρ.

2.4 Interpretation in Terms of Composite Likelihood

Four estimation methods :

• the standard one, that applies recursively the maximisation of the marginal log-likelihood : $\sum_{t=1}^{T} \log f_{1,1}(y_t; \delta)$,

and of the pairwise log-likelihood (with $h = 1$) :

$\sum_{t=1}^{T} \log f_{2,1}(y_t, y_{t-1}; \delta, \rho)$.

• the composite likelihood method (CPML1), that jointly maximizes :

$$\sum_{t=1}^{T} \log f_1(y_t; \delta) + \sum_{t=1}^{T} \log f_{2,1}(y_t, y_{t-1}; \delta, \rho).$$

- the CPML adjusted by indirect inference to adjust for bias (since the pairwise and composite likelihoods are not exact likelihood functions)
- the composite likelihood method (CPML2) including two lags.

$$\sum_{t=1}^{T} \log f_1(y_t;\delta) + \sum_{h=1}^{2} f_{2,h}(y_t, y_{t-h};\delta,\rho^h).$$

2.5 A Monte-Carlo Study

$T = 100, \delta = 0, \rho = 0, 0.25, 0.50, 0.75.$

	R		CPML1		CII		CPML2	
	δ	ρ	δ	ρ	δ	ρ	δ	ρ
$\rho = 0$	.124	.151	.123	.151	.123	.148	.122	.148
$\rho = .25$	.150	.150	.149	.150	.149	.149	.148	.149
$\rho = .50$	.203	.142	.202	.141	.202	.140	.201	.136
$\rho = .75$	.286	.116	.285	.115	.285	.114	.284	.109

TABLE 1 : RMSE of the estimators of $\delta (\delta = 0)$ and ρ.

An improvement of about 5% on the estimated ρ, when increasing the number of lags.

3. EXTENSIONS

Two extensions are considered :

- The AR(1) model with non Gaussian innovation.
- A dynamic latent model with autoregressive systematic risk factor.

3.1 NonGaussian Latent Model

The latent model becomes :

$$y_t^* = m + \rho(y_{t-1}^* - m) + u_t^*,$$

where u_t^* is i.i.d.

Closed form expression when u_t^* is Gaussian, or when u_t^* is Cauchy.
The marginal and pairwise likelihoods corresponding to Gaussian, or Cauchy can be used, even if u_t^* is neither Gaussian, nor Cauchy. But the bias has to be adjusted for by indirect inference.

Monte-Carlo : $m = 0, \rho = 0.75, u_t^* \sim$ Student with 8 degrees of freedom and unit variance.

We use a Gaussian composite likelihood CPML2, then bias adjusted by indirect inference. The RMSE are the following ones :

	CPML2	CII
δ	.301	.276
ρ	.113	.111

The RMSE is larger than in Table 1 due to the misspecification and the improvement by indirect inference is more important especially for parameter δ.

3.2 Systematic Risk

- The latent model :

$$\begin{cases} y_{it}^* = m + \gamma F_t + \sqrt{1-\gamma^2} u_{it}^*, i = 1, \ldots, n, t = 1, \ldots, T, \\ F_t = \rho F_{t-1} + \sqrt{1-\rho^2} v_t, \end{cases}$$

where the (u_{it}^*) are i.i.d., the $v_t's$ are i.i.d. and independent of the $u_{it}^{*'} s$.

- the observable variables :

$$y_{i,t} = 1, \text{ if } y_{i,t}^* > 0, y_{i,t} = 0, \text{ otherwise.}$$

The likelihood function involves T dimensional integrals and is untractable.

i) The Standard Approach

Under the Gaussian assumption on $u*$, we have :
the conditional (PIT) probability of default :

$$PD_t = P[y_{it} = 1|F_t] = \Phi\left(\frac{m + \gamma F_t}{\sqrt{1-\gamma^2}}\right),$$

or approximately :

$$\Phi^{-1}(\widehat{PD}_t) \simeq \frac{m}{\sqrt{1-\gamma^2}} + \frac{\gamma}{\sqrt{1-\gamma^2}} F_t.$$

The factor is defined up to drift and scale. Thus the normalized smoothed factor is derived by demeaning and standardizing the observed $\Phi^{-1}(\widehat{PD}_t)$.

Then :

$$\hat{F}_t = \frac{\Phi^{-1}(\widehat{PD}_t) - \hat{\mu}}{\hat{\sigma}},$$

where $\hat{\mu}, \hat{\sigma}^2$ are the historical mean and variance of $\Phi^{-1}(\widehat{PD}_t)$.

The estimators of m and γ^2 are deduced by :

$$\hat{\gamma}^2 = \frac{\hat{\sigma}^2}{1+\hat{\sigma}^2}, \hat{m} = \frac{\hat{\mu}}{(1+\hat{\sigma}^2)^{1/2}}.$$

[see Vasicek (19991)].

ii) Composite Pseudo-Likelihood

This approach considers the maximization of :

$$\sum_{t=1}^{T}\sum_{i=1}^{n}\log f_2^s(y_{it}, y_{i,t-1}; m, \gamma, \rho) + \sum_{t=1}^{T}\sum_{i<j}\log f_2^w(y_{it}, y_{jt}; m, \sigma),$$

where : f^s is the serial pairwise likelihood,

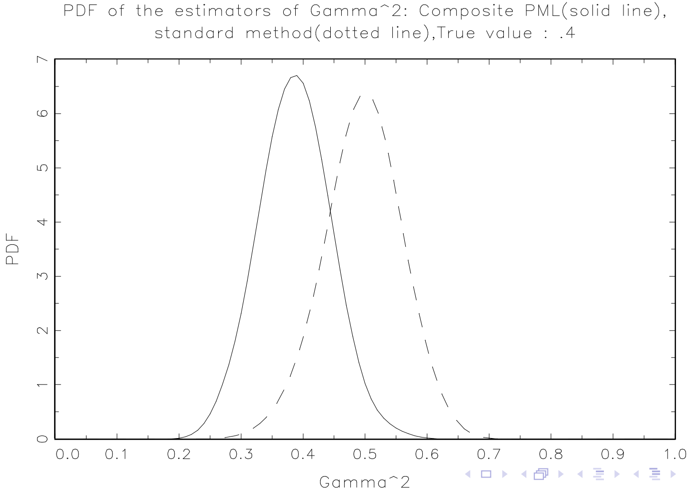
f^w is the within pairwise likelihood.

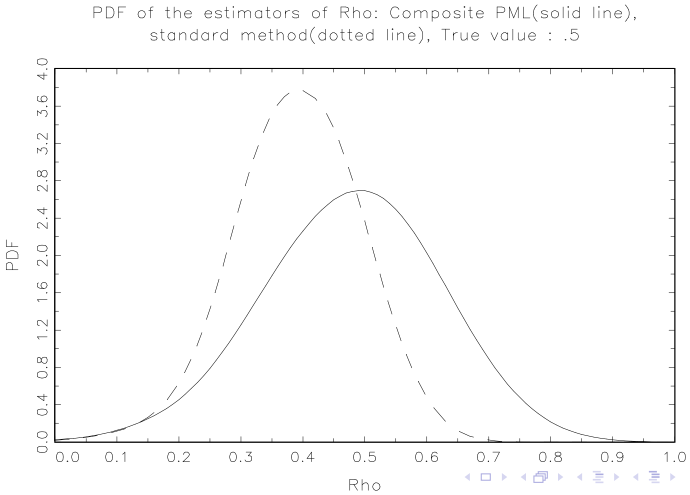
There is a closed form, if the $v's$ are also Gaussian

- Monte-Carlo exercise :
$m = 0, \gamma^2 = 0.4, \rho = 0.5, T = 100, n = 20, 50, 100$

n		m	γ^2	ρ
20	S	.114	.111	.141
	C	.111	.053	.130
50	S	.114	.058	.108
	C	.111	.048	.108
100	S	.114	.050	.102
	C	.111	.050	.102

TABLE 2 : RMSE of the Standard (S) and Composite PML (C) estimators


PDF of the estimators of Gamma^2: Composite PML(solid line),
standard method(dotted line),True value : .4
PDF
Gamma^2
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
0
1
2
3
4
5
6
7


PDF of the estimators of Rho: Composite PML(solid line),
standard method(dotted line), True value : .5
PDF
0.0
0.4
0.8
1.2
1.6
2.0
2.4
2.8
3.2
3.6
4.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Rho

4. CONCLUSION

The aim of this paper was to illustrate the implementation of the composite indirect inference approach in complicated nonlinear dynamic factors, as the models encountered in credit risk analysis.