Alpha-CIR Model in Sovereign Interest Rate Modelling

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Motivation

- Current sovereign bond markets in the Euro zone:
  - persistency of low interest rates
  - significant fluctuations at local extent.

Figure: Long term interest rates of Euro area countries.
Modelling approaches

- Large fluctuations in financial data motivate the introduction of jumps in the interest rate dynamics: Eberlein & Raible (1999), Filipović, Tappe & Teichmann (2010)...


- Difficulty: jump presence v.s. trend of low interest rate
Plan of our work

- Objective: a new model of interest rate ($\alpha$-CIR model) for these seemingly puzzling phenomena in a unified and parsimonious framework.
- Jump diffusion model as natural extension of the CIR model, using the $\alpha$-stable branching processes
  - CIR model is the particular case with continuous path
- Integral representation to highlight the branching property, Dawson and Li (2006):
  - limit of Hawkes processes: clustering and self-exciting properties;
  - link with CBI processes: exponential affine structure for bond price, Duffie, Filipović & Schachermayer (2001)
- The bond price decreases with the parameter $\alpha$, which allows to respond to the low interest rate behavior.
The $\alpha$-CIR model setup

We consider $\alpha$-CIR($a, b, \sigma, \sigma_Z, \alpha$) model for the short interest rate

$$r_t = r_0 + \int_0^t a (b - r_s) \, ds + \sigma \int_0^t \sqrt{r_s} \, dB_s + \sigma_Z \int_0^t r_s^{1/\alpha} \, dZ_s \quad (1)$$

- $B = (B_t, t \geq 0)$ a Brownian motion
- $Z = (Z_t, t \geq 0)$ a spectrally positive $\alpha$-stable compensate Lévy process with parameter $\alpha \in (1, 2]$ with

$$\mathbb{E} \left[ e^{-qZ_t} \right] = \exp \left\{ - \frac{tq^\alpha}{\cos(\pi \alpha/2)} \right\}, \quad q \geq 0.$$  

- $B$ and $Z$ are independent

$Z_t$ follows the $\alpha$-stable distribution $S_\alpha(t^{1/\alpha}, 1, 0)$ with scale parameter $t^{1/\alpha}$, skewness parameter 1 and zero drift.
A natural extension of the CIR model

- Existence of the unique strong solution by Fu and Li (2010).
- When $\sigma_Z = 0$, we recover the CIR model.
- When $\alpha = 2$, it also reduces to a CIR model but with volatility parameter $(\sigma^2 + 2\sigma_Z^2)^{1/2}$.

Figure: Lévy process $Z$ and the corresponding rate $r$ with different $\alpha$. 
The difference of $Z$ from a Brownian motion is controlled by the tail index $\alpha$:

- $\alpha = 2$: $Z$ is a Brownian motion scaled by $\sqrt{2}$;
- $\alpha < 2$: $Z$ is a pure jump process with heavy tails. More as $\alpha$ close to 1, more likely $Z_t$ takes values far from median;
- $1 < \alpha < 2$: $Z$ is a pure jump process with infinite-variation: comparison with Poisson process, $Z$ has an infinite number of (small) jumps over any time interval, allowing it to capture the extreme activity.
- A smaller $\alpha$ is related to a deeper (negative) compensation of $Z$. 
Similar properties with CIR model

Boundary condition:
The point 0 is an inaccessible boundary if and only if $2ab \geq \sigma^2$. In particular, a pure jump $\alpha$-CIR process with $ab > 0$ never reaches 0 since $\sigma = 0$.

Branching property:
$r$ can be decomposed as $r = r^{(1)} + r^{(2)}$ where for $i = 1, 2$, $r^{(i)}$ is an $\alpha$-CIR($a, b^{(i)}, \sigma, \sigma_Z, \alpha$) process such that $r_0 = r_0^{(1)} + r_0^{(2)}$ and $b = b^{(1)} + b^{(2)}$. 
Integral representation

Integral form by using the random fields

\[ r_t = r_0 + \int_0^t a(b - r_s) \, ds + \sigma \int_0^t \int_0^{r_s} W(ds, du) \]
\[ + \sigma Z \int_0^t \int_0^{r_s-} \int_{\mathbb{R}^+} \zeta \tilde{N}(ds, du, d\zeta), \]  

- \( W(ds, du) \): white noise on \( \mathbb{R}^2_+ \) with intensity \( dsdu \),
- \( \tilde{N}(ds, du, d\zeta) \): compensated Poisson random measure on \( \mathbb{R}^3_+ \) with intensity \( dsdu\mu(d\zeta) \),
- \( \mu(d\zeta) \) is a Lévy measure satisfying \( \int_0^\infty (\zeta \wedge \zeta^2) \mu(d\zeta) < \infty \).

Equivalence of two representations

We choose the Lévy measure to be

\[ \mu(d\zeta) = -\frac{1\{\zeta>0\}d\zeta}{\cos(\pi\alpha/2)\Gamma(-\alpha)\zeta^{1+\alpha}}, \quad 1 < \alpha < 2, \]

Then the root representation (1) and the integral representation (2) are equivalent in the following sense by Li (2011):

- The solutions of the two equations have the same probability law.
- On an extended probability space, they are equal almost surely.
Link to Hawkes process

- When $\sigma = 0$ and $\mu(d\zeta) = \delta_1(dz)$, then $r$ is given by

$$r_t = r_0 + abt - \int_0^t (a + \sigma Z) r_s ds + \sigma Z \int_0^t \int_0^{r_s-} N(ds, du) \tag{4}$$

which is the intensity of Hawkes process $\int_0^t \int_0^{r_s-} N(ds, du)$, $N$ being the Poisson random measure with intensity $dsdu$.

- Consider a sequence $\{r_t^{(n)}, t \geq 0\}$ defined by (4) with parameters $(a/n, nb, \sigma Z)$. Then

$$r_{nt}^{(n)}/n \xrightarrow{L} Y_t \quad \text{in} \ D(\mathbb{R}_+),$$

where $D(\mathbb{R}_+)$ is the Skorokhod space of càdlàg processes and

$$Y_t = \int_0^t a(b - Y_s) ds + \sigma Z \int_0^t \int_0^{Y_s} W(ds, du).$$

Locally equivalent Lévy-Ornstein-Uhlenbeck process

- Consider the $\alpha$-CIR process with initial value $r_0$ and introduce

$$\lambda_t = r_0 + \int_0^t a(b - \lambda_s) \, ds + \sigma \int_0^t \int_0^{r_0} W(ds, du) + \sigma Z \int_0^t \int_0^{r_0} \int_{\mathbb{R}^+} \zeta \tilde{N}(ds, du, d\zeta)$$

where the processes $W$ and $\tilde{N}$ are the same as in (1).

- The above LOU process can be written as

$$\lambda_t = r_0 + \int_0^t a(b - \lambda_s) \, ds + \sigma \sqrt{r_0} B_t + \sigma Z \sqrt{r_0} Z_t,$$

- The implicit negative drifts lead to a linear decay for $\lambda_t$ while an exponential decay for $r_t$: when $\sigma_Z$ increases, the decreasing drift plays a more important role in $\alpha$-CIR than in LOU.
Comparison between $\alpha$-CIR and LOU (continued)

- Separating small and large jumps in LOU, we get

$$\lambda_t = r_0 + \int_0^t a\left(b - \frac{\sigma Z r_0 \Theta(\alpha, y)}{a} - \lambda_s\right)ds + \sigma \int_0^t \int_0^{r_0} W(ds, du)$$

$$+ \sigma Z \int_0^t \int_0^{r_0} \int_0^y \zeta \tilde{N}(ds, du, d\zeta) + \sigma Z \int_0^t \int_0^{r_0} \int_y^{\infty} \zeta N(ds, du, d\zeta)$$

where

$$\Theta(\alpha, y) = \frac{2}{\pi} \alpha \Gamma(\alpha - 1) \frac{\sin(\pi \alpha/2)}{y^{\alpha - 1}}.$$

- In a similar way, the $\alpha$-CIR process can be written as

$$r_t = r_0 + \int_0^t \tilde{a}(\alpha, y)\left(\tilde{b}(\alpha, y) - r_s\right)ds + \sigma \int_0^t \int_0^{r_s} W(ds, du)$$

$$+ \sigma Z \int_0^t \int_0^{r_s} \int_0^y \zeta \tilde{N}(ds, du, d\zeta) + \sigma Z \int_0^t \int_0^{r_s} \int_y^{\infty} \zeta N(ds, du, d\zeta)$$

where

$$\tilde{a}(\alpha, y) = a + \sigma Z \Theta(\alpha, y), \quad \tilde{b}(\alpha, y) = \frac{ab}{a + \sigma Z \Theta(\alpha, y)}.$$
Continuous state branching process with immigration (CBI)

CBI (Kawazu & Watanabe 1971) of branching mechanism $\Psi(\cdot)$ and immigration rate $\Phi(\cdot)$: Markov process $X$ with state space $\mathbb{R}_+$ verifying

$$
\mathbb{E}_x \left[ e^{-pX_t} \right] = \exp \left[ -x\nu(t, p) - \int_0^t \Phi(\nu(s, p)) \, ds \right],
$$

where $\nu : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$
\frac{\partial \nu(t, p)}{\partial t} = -\Psi(\nu(t, p)), \quad \nu(0, p) = p
$$

and $\Psi$ and $\Phi$ are functions on $\mathbb{R}_+$ given by

$$
\Psi(q) = \beta q + \frac{1}{2} \sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu) \pi(du),
$$

$$
\Phi(q) = \gamma q + \int_0^\infty (1 - e^{-qu}) \nu(du),
$$

with $\sigma, \gamma \geq 0$, $\beta \in \mathbb{R}$ and $\pi, \nu$ being two Lévy measures such that $\int_0^\infty (u \wedge u^2) \pi(du) < \infty$ and $\int_0^\infty (1 \wedge u) \nu(du) < \infty$. 
Link with the CBI processes

Let $r$ be an $\alpha$-CIR $(a, b, \sigma, \sigma_Z, \alpha)$ process. Then $r$ is a CBI with
branching mechanism: $\Psi(q) = aq + \frac{\sigma^2}{2} q^2 - \frac{\sigma_Z^\alpha}{\cos(\pi\alpha/2)} q^\alpha$ (6)

immigration rate: $\Phi(q) = abq$. (7)

Consequences:

- As $t \to +\infty$, $r_t$ has a limite distribution $r_\infty$, given by
  $E[e^{-pr_\infty}] = \exp\left\{ - \int_0^p \frac{\Phi(q)}{\psi(q)} dq \right\}$, $p \geq 0$.

- Laplace transform
  $E[e^{-r_t - p \int_0^t r_s ds}] = \exp\left( - r_0 \nu(t, \xi, p) - \int_0^t \Phi(\nu(s, \xi, p)) ds \right)$,
  with $\partial_t \nu(t, \xi, p) = -\Psi(\nu(t, \xi, p)) + p$, $\nu(0, \xi, p) = \xi$.

- Let $r^{(\alpha)}$ be $\alpha$-CIR$(a, b, \sigma, \sigma_Z, \alpha)$ process, $\alpha \in (1, 2]$. Then
  $r^{(\alpha)} \xrightarrow{L} r^{(2)}$ in $D(\mathbb{R}_+)$ as $\alpha \to 2$. 

Equivalent martingale measure for bond pricing

- Let $r$ be an $\alpha$-CIR($a, b, \sigma, \sigma_Z, \alpha$) processes under the initial probability $\mathbb{P}$.
- Fix $\eta \in \mathbb{R}$ and $\theta \in \mathbb{R}_+$, and define
  \[ U_t := \eta \int_0^t \int_0^{r_s} W(ds, du) + \int_0^t \int_0^{r_s^-} \int_0^\infty (e^{-\theta \zeta} - 1) \tilde{N}(ds, du, d\zeta). \]
- Change of probability: $\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(U)$, with $\mathcal{E}(U)$ the Doléans-Dade exponential of $U$ (Kallsen & Muhle-Karbe, 2010).
- $r$ is an $\alpha$-CIR($a', b', \sigma, \sigma_Z, \alpha$) type process under $\mathbb{Q}$ with
  \[ a' = a - \sigma \eta - \frac{\alpha \sigma Z}{\cos(\pi \alpha/2)} \theta^{\alpha-1}, \quad b' = ab/a', \]
  and a modified Lévy measure
  \[ \mu'(d\zeta) = -\frac{e^{-\theta \zeta} 1_{\{\zeta > 0\}}}{\cos(\pi \alpha/2) \Gamma(-\alpha) \zeta^{1+\alpha}} d\zeta. \]
- $r$ remains to be a CBI process under $\mathbb{Q}$. 
Application to bond pricing

For simplicity, we assume that the short rate $r$ is given by an $\alpha$-CIR($a$, $b$, $\sigma$, $\sigma_Z$, $\mu$, $\alpha$) model under $\mathbb{Q}$.

- Zero-coupon bond price:

$$B(t, T) = \exp \left( - r_t \nu(T - t) - ab \int_0^{T-t} \nu(s) ds \right),$$

where $\nu(\cdot)$ is given by

$$\frac{\partial \nu(t)}{\partial t} = 1 - \Psi(\nu(t)), \quad \nu(0) = 0,$$

with $\Psi(q) = aq + \frac{\sigma^2}{2} q^2 - \frac{\sigma_Z^x}{\cos(\pi \alpha/2)} q^\alpha$.

- We have

$$\nu(t) = f^{-1}(t) \text{ where } f(t) = \int_0^t \frac{dx}{1 - \Psi(x)} \quad (8)$$
Proposition
The function $\nu(\cdot)$ is increasing with respect to $\alpha \in (1, 2]$. In particular, the bond price $B(0, T)$ is decreasing with respect to $\alpha$.

- $\alpha$ characterizes the tail fatness: when $\alpha$ decreases, it is more likely to take values far away from median and have large jumps.
- Generalized Blumenthal-Getoor index (e.g. Aït-Sahalia and Jacod, 2009) $\inf\{\beta > 0 : \sum_{0 \leq s \leq T} \Delta r_s^\beta < \infty, \ a.s.\} = \alpha$.
- The above proposition shows that the $\alpha$-CIR model is suitable to describe the phenomenon of low interest rate trend with jumps.
An explanation of Proposition

- The compensated $\alpha$-stable Lévy process $Z$ in the $\alpha$-CIR model: a smaller $\alpha$ is related to a deeper (negative) compensation and hence a stronger mean-reversion.

- As the interest rate becomes low because of the mean-reversion effect, the self-exciting property will imply a decreasing frequency of jumps and enforce the tendency of low interest rate.
Simulation of processes $Z$ and $r$ with different $\alpha$

Figure: Three parameters of $\alpha$: 2 (blue), 1.5 (green) and 1.2 (black)
Figure: Bond price is decreasing w.r.t. $\alpha$, curve CIR (in red) corresponds to $\sigma_Z = 0$. 

$r_0 = 0.05$, $a = 0.1$, $b = 0.3$, $\sigma = 0.1$, $\sigma_Z = 0.3$
Jump behavior

- The jumps, especially the large jumps capture the significant changes in the interest rate and may imply the downgrade risk of credit quality.
- Fix $y > 0$ and define the first time that the jump of $r$ is large than $\sigma_Z y$, i.e. $\tau_y = \inf\{t > 0 : \Delta r_t > \sigma_Z y\}$.
- Consider the truncated process $r^{(y)}$ as

$$r_t^{(y)} = r_0 + \int_0^t \bar{a}(\alpha, y)(\bar{b}(\alpha, y) - r_s)ds + \sigma \int_0^t \int_0^{r_s} W(ds, du)$$

$$+ \sigma_Z \int_0^t \int_0^{r_s} \int_0^y \zeta \tilde{N}(ds, du, d\zeta).$$

- It is also a CBI process which coincides with $r$ up to $\tau_y$, and with the branching mechanism given by

$$\psi^{(y)} = \psi + \sigma_Z^{\alpha} \int_y^{\infty} (1 - e^{-q\zeta}) \mu(d\zeta).$$
Probability law of the first large jump

We have

\[ \mathbb{P}(\tau_y > t) = \exp \left( -l(y, t) r_0 - ab \int_0^t l(y, s) ds \right) \]

where \( l(y, t) \) is the unique solution of

\[ \frac{dl}{dt}(y, t) = \sigma_Z^\alpha \int_y^\infty \mu(d\zeta) - \Psi(y)(l(y, t)), \]

with initial condition \( l(y, 0) = 0. \)

- Equivalent form:

\[ \mathbb{P}(\tau_y > t) = \mathbb{E} \left[ \exp \left\{ -\sigma_Z^\alpha \left( \int_y^\infty \mu(d\zeta) \left( \int_0^t r_s^{(y)} ds \right) \right) \right\} \right]. \]

which is a bond price written on the auxiliary rate \( r^{(y)} \) weighted by the measure \( \mu \) restricted on \((y, \infty)\).
Probability function \( \mathbb{P}(\tau_y > t) \) for the first big jump and the expectation of \( \tau_y \)

**Figure:** Probability function \( \mathbb{P}(\tau_y > t) \) and expectation of the first jump time \( \tau_y \) of the short rate \( r \) whose jump size is larger than \( y \).
Thanks for your attention!