Return Predictability and Risk Management

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Background

Good risk management

- Important for regulators and financial institutions
- E.g. Basel II: one component of capital requirement based on 10-day VaR at 1%
  - If overestimate, too much capital held $\rightarrow$ bad for economy
  - If underestimate, could cause systemic risk
- Other measures: Expected shortfall (ES), CoVar, SRISK, etc.

Background

Common way to compute risk measures

- Model the process of daily return, such as

$$y_t = \mu_{t-1} + \sigma_{t-1} u_t, \quad u_t \sim i.i.d.(0, 1)$$

- Well known that μ_{t-1} hard to detect at daily level
  $\rightarrow$ often assumed to be constant (μ)
- Christoffersen 2011, SRISK (Brownlees-Engle, 2016)

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From literature on return predictability

- Time-varying cond. mean = predictability of asset returns
  - Well documented in the literature (e.g. dividend-price ratio)
  - Especially for long-horizons

This paper

Question

Q1: Assuming μ instead of μ_t is problematic to compute risk measures?

- For now, focus on VaR

Q2: If yes, what is the better way than assuming μ?

Methodology

- With known DGP (2 scenarios)
  - Q1: Yes
  - Q2: Even if $H_0 : \mu_t$ is constant cannot be rejected, estimate with μ_t and compute VaR is in expectation better.
- Empirical studies (w/ unknown DGP)
  - VaR computed assuming w/ and w/o μ_t makes difference
  - Cannot determine which is better...
- (from now on) With unknown DGP?

With known DGP

DGP: Two scenarios

1: Conditional volatility model, $y_t = \mu_{t-1} + \sigma u_t$

2: GARCH-in-Mean type model, $y_t = r + \lambda h_{t-1} + \sqrt{h_{t-1}} u_t$

Two types of agents

- ("Correct" agent) Use the "correct" model
- ("Naive" agent) Simplify the model and assume μ
- Compare two VaR computed by correct and naive agents

Procedure

- For each scenario, define "misspecified" model (naive agent).
- w/ known "true" parameter values, compute VaR
- Monte-Carlo simulation to study when pars are estimated

Roadmap

1. Introduction
2. Constant Volatility Model
   - Correct model and Misspecified model
   - "True" parameters
   - VaR without parameter uncertainty
   - VaR with parameter uncertainty (Monte Carlo)
3. Garch-in-Mean type Model (Heston-Nandi)
4. Empirical studies
5. Conclusion

Constant Volatility Model

Consider some asset with price at time t, P_t.
Define $y_t = \log P_t - \log P_{t-1}$
Frequency is daily.

Correct Model

$$
\begin{aligned}
y_t &= \mu_{t-1} + u_t \\
\mu_t &= c + \theta \mu_{t-1} + w_t \\
(u_t, w_t)' &\sim i.i.d.\ N\left(0, \begin{pmatrix} \sigma_u^2 & \sigma_{uw} \\ \sigma_{uw} & \sigma_w^2 \end{pmatrix}\right)
\end{aligned}
$$

- Widely used in return predictability literature

  Campbell(01), Barberis(00), Pastor & Stambaugh(09),etc
- $|\theta| < 1$

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\end{aligned}
$$

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Misspecified Model

$$
\begin{aligned}
y_t &= \mu_y + \tilde{u}_t \\
\tilde{u}_t &\sim i.i.d.\ N(0, \sigma_y^2)
\end{aligned}
$$

where $\mu_y = E(y_t)$, $Var(y_t) = \sigma_y^2$.

"True" parameter values

	Benchmark
θ	0.999
σ_μ^2	0.0003 $\times \sigma_y^2$
$corr(u_t, w_t)$	-0.8
μ_y	Empirical mean of y_t (S&P 500 index)
σ_y^2	Empirical variance of y_t

Why these values? *to match stylized facts*

- θ close to 1
  - Expected return very persistent
  - $\rightarrow$ *Evidence of return predictability at long horizons*
- $\sigma_\mu^2 / \sigma_y^2$ very low ("R^2")
  - Since $\sigma_y^2 = \sigma_\mu^2 + \sigma_u^2$, $\sigma_u^2 >> \sigma_\mu^2$
  - Variation in y_t mostly comes from variation in u_t
  - $\rightarrow$ *Hard to detect μ_t when looking at data*
- $corr(u_t, w_t) < 0$ More
- Consistent with Pastor-Stambaugh(12) More

Other parameter values

Plus, we vary the par. values fixing others fixed:

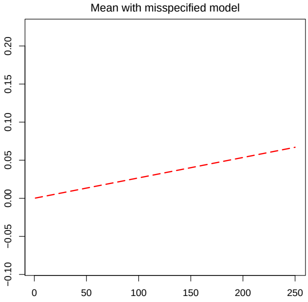
- θ: 0.99, 0.98, 0.95
- σ_μ^2: 0.00005, 0.0005, 0.001 $\times \sigma_y^2$

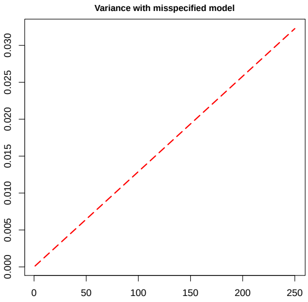
		$R^2 = \sigma_\mu^2/\sigma_y^2$			
		0.00005	0.0003	0.0005	0.001
	0.999	x	Benchmark	x	x
θ	0.99		x		
	0.98		x		
	0.95		x		

Naive agent's forecast

Define $y_{t:t+\tau} = \sum_{j=1}^{\tau} y_{t+j}$, τ-day ahead log-return.

$$
\begin{aligned}
y_{t:t+\tau} | M_{miss} &\sim N(E_{miss},\ V_{miss}) \\
E_{miss} &= \mu_y \tau \\
V_{miss} &= \sigma_y^2 \tau
\end{aligned}
$$





Forecast of agent with Correct model + observed μ_t

$$y_{t:t+\tau} | \underline{\mu_t}, \underline{y_t} \quad \sim \quad N(E_{correct}, \; V_{correct})$$

$$\begin{aligned} E_{correct} &= \mu_y \tau + (\mu_t - \mu_y) \frac{1 - \theta^\tau}{1 - \theta} \\ V_{correct} &= \tau \left(\sigma_u^2 + \frac{2\sigma_{uw}}{1 - \theta} A_\tau + \frac{\sigma_w^2}{(1 - \theta)^2} B_\tau \right) \end{aligned}$$

A_τ and B_τ both $\nearrow$ in τ, between 0 and 1

$$A_\tau = 1 - \frac{1 - \theta^\tau}{\tau(1 - \theta)}$$

$$B_\tau = 1 - \frac{2(1 - \theta^\tau)}{\tau(1 - \theta)} + \frac{1 - \theta^{2\tau}}{\tau(1 - \theta^2)}$$

Forecast of agent with Correct model + observed μ_t

$$y_{t:t+\tau} | \underline{\mu_t}, \underline{y_t} \quad \sim \quad N(E_{correct},\ V_{correct})$$

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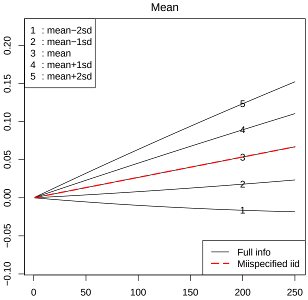
$$B_\tau = 1 - \frac{2(1-\theta^\tau)}{\tau(1-\theta)} + \frac{1-\theta^{2\tau}}{\tau(1-\theta^2)}$$

Mean

$$E_{correct} = \mu_y \tau + (\mu_t - \mu_y) \frac{1 - \theta^\tau}{1 - \theta}$$
$$E_{miss} = \mu_y \tau$$

- If $\mu_t = \mu_y$, $E_{correct} = E_{miss}$
- If $\mu_t > \mu_y$, $E_{correct} > E_{miss}$

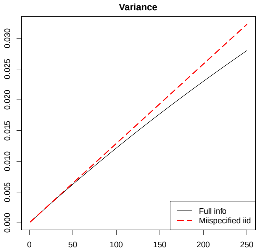


Variance

$$V_{correct} = \tau \left(\sigma_u^2 + \frac{2\sigma_{uw}}{1-\theta} A_\tau + \frac{\sigma_w^2}{(1-\theta)^2} B_\tau \right)$$

$$V_{miss} = \tau \sigma_y^2 = \tau \left(\sigma_u^2 + \frac{\sigma_w^2}{1-\theta^2} \right)$$

- NOT depend on realization of μ_t
- $V_{correct} < V_{miss}$ with our calibration where $\sigma_u^2 >> \sigma_w^2$



Value-at-risk

Will study 1% VaR in dollar term, i.e.,

$$Prob(\$Loss \geq VaR) = 0.01$$

When

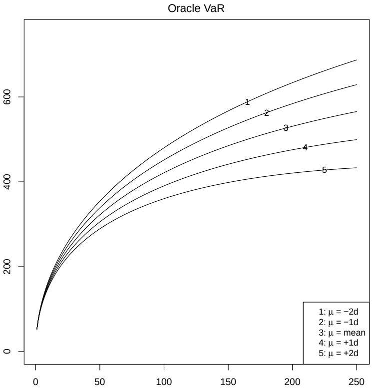
$$y_{t:t+\tau}|I_t \sim N(E, V),$$

$$VaR = (\$) \times (1 - \exp(E - 2.33\sqrt{V}))$$

- Bank of America reports one-day VaR (1%) as \$ 53M (2015).
  $\rightarrow$ Put portfolio value so that one-day VaR $\approx$ \$ 53M.
- *Mean effect*: Larger E, smaller VaR
- *Variance effect*: Larger V, larger VaR

Oracle VaR and Mean effect

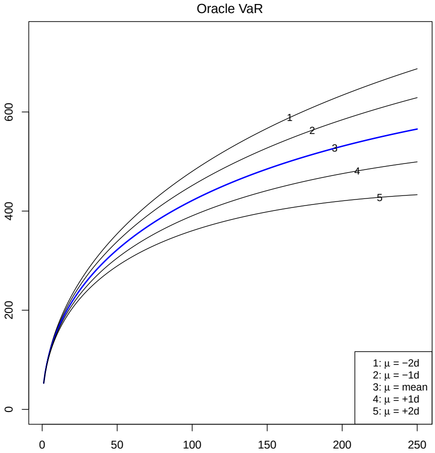
$$VaR_{correct} \quad = \quad (\$) \times (1 - \exp(E_{correct} - 2.33\sqrt{V_{correct}}))$$



- Differences in five $VaR_{correct}$ derived from $E_{correct}$

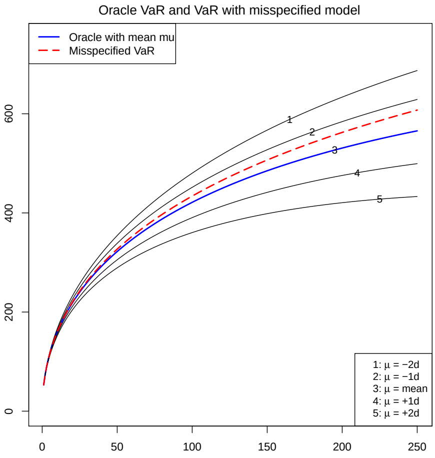
Oracle VaR when $\mu_t = \mu_y$

$$VaR_{correct} = (\$) \times (1 - \exp(\mu_y \tau - 2.33\sqrt{V_{correct}}))$$



Oracle VaR v.s. VaR with misspecidied model

$$VaR_{correct} = (\$) \times (1 - \exp(\mu_y \tau - 2.33\sqrt{V_{correct}}))$$
$$VaR_{miss} = (\$) \times (1 - \exp(\mu_y \tau - 2.33\sqrt{V_{miss}}))$$



Value-at-risk

Days	Oracle VaR					VaR_{miss}
	μ_1 -2sd	μ_2 -1sd	μ_3 mean	μ_4 +1sd	μ_5 +2sd	
1	53.8	53.4	53.0	52.6	52.2	53.0
10	166.3	162.7	159.0	155.3	151.8	159.5
22	242.4	234.9	227.1	219.3	211.7	228.6
66	401.1	381.1	359.9	338.5	317.7	367.1
125	526.8	492.6	456.1	418.7	382.0	473.1
250	687.2	628.9	565.5	499.4	433.1	607.5

million dollars

More differences between Oracle VaR and VaR_{miss} with

- More μ_t deviates from μ_y
- Longer horizon

Parameter uncertainty: simulation results

- In reality, we do not observe μ_t. What can correct agent do?
- One solution: ARMA(1,1) representation
- Filtering μ_t by $\{y_\tau, \tau \leq t\}$

Given correct model, y_t can be written as

$$\begin{aligned} y_t &= c + \theta y_{t-1} + \eta_t + \gamma \eta_{t-1} \\ \eta_t &= i.i.d.\ N(0, \sigma_\eta^2) \end{aligned}$$

- With benchmark par. values, $\theta = 0.999$, $\gamma = -0.99939$.
- Simulate 1000 different paths based on the correct model
- (Correct Agent) Estimate ARMA(1,1)
- (Naive Agent) Estimate Misspecified i.i.d. model
- Compute VaR with estimated parameters of two Agents

Result 1

Correct Agent estimates ARMA(1,1) model, i.e.,

$$\begin{aligned} y_t &= c + \theta y_{t-1} + \eta_t + \gamma \eta_{t-1} \\ \eta_t &= i.i.d.\ N(0, \sigma_\eta^2) \end{aligned}$$

For each simulation, we test

$$H_0 : y_t \sim i.i.d. N(\mu_y, \sigma_y^2)$$

- Likelihood-Ratio test at 5% level
- Fail to reject H_0 880 times out of 1000.
- Given DGP, we tend to "accept" the misspecified model.

Result 2: VaR with smaller μ_t

μ_t	Horizon (Days)	Oracle	ARMA-estimate Mean	ARMA-estimate Q-5%	ARMA-estimate Q-95%	iid-estimate Mean	iid-estimate Q-5%	iid-estimate Q-95%
μ_1	1	53.8	53.8	53.0	54.6	53.0	52.2	53.8
	10	166.3	163.1	155.7	169.6	159.5	155.6	163.2
	22	242.4	233.7	219.4	248.5	228.5	221.4	235.6
	66	401.1	372.1	338.6	414.5	366.8	348.2	384.7
	125	526.8	473.3	415.2	544.1	472.4	440.8	504.1
	250	687.2	594.6	484.4	708.1	606.0	548.2	661.8
μ_2	1	53.4	53.4	52.6	54.2	53.0	52.2	53.8
	10	162.7	161.0	154.6	167.1	159.5	155.6	163.2
	22	234.9	230.2	217.4	244.1	228.5	221.4	235.6
	66	381.1	365.5	333.2	405.1	366.8	348.2	384.7
	125	492.6	464.8	402.4	532.2	472.4	440.8	504.1
	250	628.9	584.8	460.7	697.7	606.0	548.2	661.8

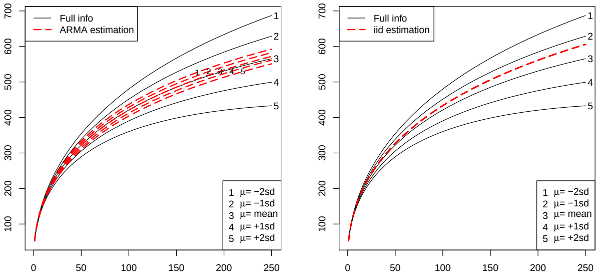
- Except for red numbers, ARMA(1,1) better

VaR with larger μ_t

μ_t	Horizon (Days)	Oracle	ARMA-estimate Mean	ARMA-estimate Q-5%	ARMA-estimate Q-95%	iid-estimate Mean	iid-estimate Q-5%	iid-estimate Q-95%
μ_3	1	53.0	53.0	52.2	53.8	53.0	52.2	53.8
	10	159.0	158.8	153.0	164.9	159.5	155.6	163.2
	22	227.1	226.5	214.0	240.1	228.5	221.4	235.6
	66	359.9	358.6	320.7	397.1	366.8	348.2	384.7
	125	456.1	455.9	384.7	524.4	472.4	440.8	504.1
	250	565.5	574.4	436.2	689.2	606.0	548.2	661.8
μ_4	1	52.6	52.6	51.9	53.4	53.0	52.2	53.8
	10	155.3	156.6	150.7	163.9	159.5	155.6	163.2
	22	219.3	222.7	209.3	237.9	228.5	221.4	235.6
	66	338.5	351.6	307.3	391.4	366.8	348.2	384.7
	125	418.7	446.9	361.5	516.0	472.4	440.8	504.1
	250	499.4	563.8	405.5	680.7	606.0	548.2	661.8
μ_5	1	52.2	52.2	51.5	53.0	53.0	52.2	53.8
	10	151.8	154.5	148.0	163.1	159.5	155.6	163.2
	22	211.7	219.1	203.2	235.9	228.5	221.4	235.6
	66	317.7	344.8	291.6	388.5	366.8	348.2	384.7
	125	382.0	438.1	339.0	509.6	472.4	440.8	504.1
	250	433.1	553.5	376.6	674.1	606.0	548.2	661.8

Results 2: Value-at-risk

Sample mean of VaR with ARMA(1,1) (left) and i.i.d (right)



- Overall, ARMA(1,1) does better job in expectation.
- ARMA(1,1) not the best one though.
  - Info on $\{y_\tau, \tau \leq t\}$ not enough to extract info. on μ_t
  - Possible solution: predictor variable
- Message: Even if ARMA(1,1) not significant, compute VaR with ARMA(1,1) better

Heston-Nandi Model

Correct Model

$$
\begin{aligned}
y_t &= \underbrace{r + \lambda h_{t-1}}_{\text{cond. mean}} + \underbrace{\sqrt{h_{t-1}}}_{\text{cond. volatility}} \quad u_t \\
u_t &\sim i.i.d.\ N(0,1) \\
h_t &= \omega + \beta h_{t-1} + \alpha(u_t - \gamma\sqrt{h_{t-1}})^2
\end{aligned}
$$

- Time-variation of cond. mean is proportional to the cond. variance. (GARCH-in-Mean)
- Affine model: cumulant of $y_{t:t+\tau}|I_t$ is affine in h_t
  $\rightarrow$ by inversion, can obtain CDF of $y_{t:t+\tau}|I_t$
  $\rightarrow$ can obtain analytical VaR for any horizon τ

Heston-Nandi Model

Correct Model

$$
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h_t &= \omega + \beta h_{t-1} + \alpha(u_t - \gamma\sqrt{h_{t-1}})^2
\end{aligned}
$$

Misspecified Model

$$
\begin{aligned}
y_t &= \underbrace{\tilde{r}}_{\text{cond. mean}} + \underbrace{\sqrt{h_{t-1}}}_{\text{cond. volatility}} u_t \\
u_t &\sim i.i.d.\ N(0,1) \\
h_t &= \omega + \beta h_{t-1} + \alpha(u_t - \gamma\sqrt{h_{t-1}})^2 \\
\tilde{r} &= r + \lambda E(h_t)
\end{aligned}
$$

“True” parameters

- Estimate the “correct model” using S&P500 Index.
- Impose $r > 0$ (risk-free rate)

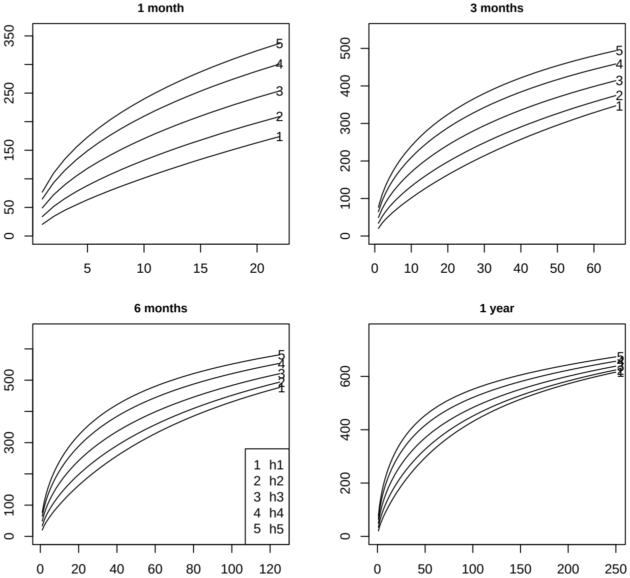
	This paper
λ	2.224
r	3.49e-09
ω	-7.03e-07
α	3.62e-06
β	0.838
γ	192.87
$E(h_t)$	1.09e-04
Obs	6553
Start	02/01/95
End	31/12/15

"True" parameters

- Estimate the "correct model" using S&P500 Index.
- Impose $r > 0$ (risk-free rate)

	This paper	HN(2000)	CJO(2009)	CJO(2011)
λ	2.224	0.205	2.899	1.661
r	3.49e-09	fixed	fixed	fixed
ω	-7.03e-07	5.02e-6	-7.756e-7	-1.269e-6
α	3.62e-06	1.32e-6	4.546e-6	2.807e-6
β	0.838	0.589	0.9041	0.9451
γ	192.87	421.39	115.9	116.0
$E(h_t)$	1.09e-04			
Obs	6553	755	2547	
Start	02/01/95	01/08/92	02/01/95	06/62
End	31/12/15	12/30/94	31/12/04	12/09

Oracle VaR

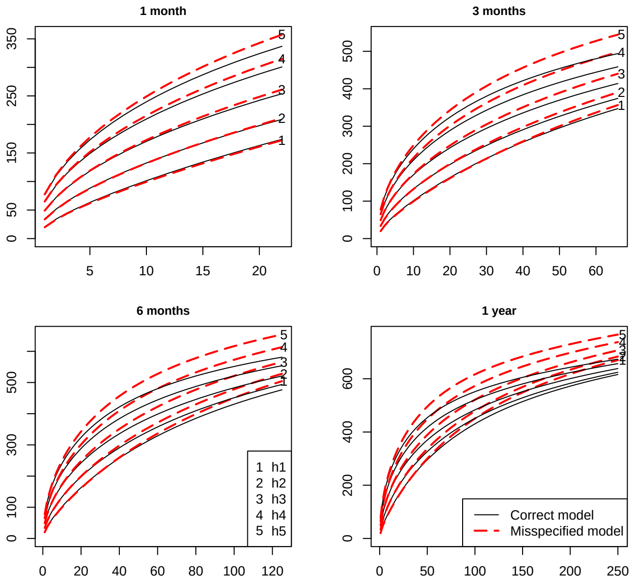


Oracle VaR

Days	h_1 A	h_2 B	h_3 C	h_4 D	h_5 E	E-A	(E-A)/C
1	20.3	33.9	49.0	64.6	76.7	56.4	1.2
10	101.4	132.5	170.3	209.5	239.7	138.3	0.8
22	173.9	208.9	253.8	300.8	337.0	163.1	0.6
66	347.0	374.4	414.0	458.6	493.9	146.9	0.4
125	476.6	493.7	520.8	553.7	581.3	104.7	0.2
250	616.0	624.4	638.4	656.9	673.4	57.5	0.1

- Difference in h_t has huge impact on VaR_{oracle}
- Difference grows until 1 month horizon, then decreases

Oracle VaR v.s. VaR with misspecified model



Oracle VaR v.s. VaR with misspecified model

h_1	Oracle	VaR_{miss}	Difference
1	20.3	19.9	-0.4
10	101.4	99.4	-2.0
22	173.9	172.2	-1.6
66	347.0	356.4	9.5
125	476.6	504.3	27.7
250	616.0	670.0	54.0

h_2	Oracle	VaR_{miss}	Difference
1	33.9	33.6	-0.2
10	132.5	132.1	-0.4
22	208.9	210.8	1.9
66	374.4	390.2	15.8
125	493.7	527.9	34.2
250	624.4	683.6	59.2

Others

h_3	Oracle	VaR_{miss}	Difference
1	49.0	49.0	0.0
10	170.3	172.6	2.3
22	253.8	261.0	7.2
66	414.0	440.2	26.1
125	520.8	566.1	45.3
250	638.4	706.7	68.3
h_4	Oracle	VaR_{miss}	Difference
1	64.6	65.0	0.4
10	209.5	215.4	5.9
22	300.8	314.8	14.1
66	458.6	497.6	39.0
125	553.7	613.4	59.6
250	656.9	737.8	80.9
h_5	Oracle	VaR_{miss}	Difference
1	76.7	77.6	0.8
10	239.7	249.0	9.3
22	337.0	357.3	20.4
66	493.9	544.5	50.5
125	581.3	653.8	72.5
250	673.4	766.1	92.7

Parameter Uncertainty: simulation results

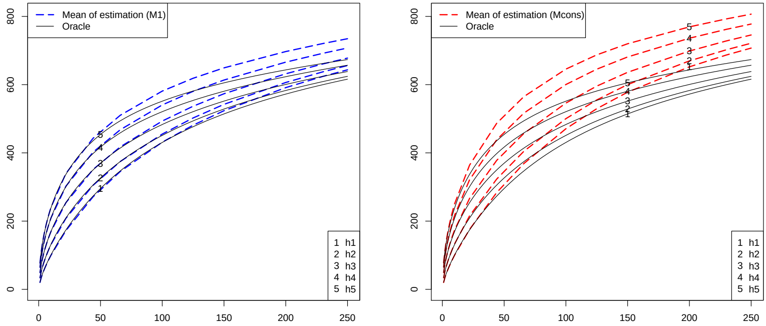
- Simulate 1000 paths from correct model
- (Correct agent) Estimates correct model
- (Naive agent) Estimates misspecified model
- Compare two VaRs based on two different estimations

Result 1

$$H_0 : \lambda = 0$$

- Likelihood-Ratio test at 5% level
- We fail to reject 860 times out of 1000.
- Confidence interval for λ
  - Lower bound: -1.52 on average
  - Upper bound: 4.34 on average

Sample mean of VaR with correct and miss. model



- Sample mean of VaR with mss. model deviates more from the oracle

VaR: smaller h_t

	Horizon (days)	Oracle	Correct			Missd		
			Mean	Q 5&	Q 95%	Mean	Q 5&	Q 95%
h_1	1	20.3	20.3	19.9	20.4	20.2	19.9	20.5
	10	101.4	100.5	96.0	105.0	100.9	96.0	105.5
	22	173.9	172.4	160.6	184.1	175.7	164.8	186.0
	66	347.0	345.9	300.2	390.3	368.8	339.7	396.4
	125	476.6	481.9	421.7	569.1	529.5	480.9	575.9
	250	616.0	644.4	596.2	740.1	707.2	635.8	740.1
h_2	1	33.9	33.8	33.4	33.9	33.8	33.4	34.1
	10	132.5	131.9	126.2	137.1	134.0	129.4	138.2
	22	208.9	208.3	193.5	222.5	215.2	205.1	224.7
	66	374.4	375.1	321.4	427.8	404.7	376.9	431.5
	125	493.7	503.0	446.6	598.7	555.1	508.1	600.1
	250	624.4	657.0	610.6	753.0	721.4	650.7	753.0

VaR: larger h_t

	Horizon (days)	Oracle	Correct			Missd		
			Mean	Q 5&	Q 95%	Mean	Q 5&	Q 95%
h_3	1	49.0	48.9	48.4	49.3	49.1	48.8	49.5
	10	170.3	170.3	161.8	178.4	174.9	170.7	178.9
	22	253.8	254.2	234.0	274.4	266.3	256.9	275.2
	66	414.0	418.0	354.2	484.7	457.0	430.3	482.4
	125	520.8	537.5	486.8	641.5	595.7	549.3	641.0
	250	638.4	678.7	634.9	775.0	745.6	676.9	775.0
h_4	1	64.6	64.6	63.7	65.5	65.1	64.7	65.5
	10	209.5	210.2	198.0	222.5	218.0	213.7	221.8
	22	300.8	302.5	274.7	331.7	320.9	311.8	329.8
	66	458.6	468.2	414.7	548.3	516.6	490.3	543.3
	125	553.7	581.2	536.6	686.4	645.3	599.9	686.4
	250	656.9	708.2	667.7	804.5	777.6	710.6	804.5
h_5	1	76.7	76.8	75.6	78.2	77.7	77.4	78.0
	10	239.7	241.0	225.4	257.7	251.8	247.7	255.6
	22	337.0	339.9	305.3	377.3	363.9	354.9	372.9
	66	493.9	510.6	464.0	601.8	564.9	538.2	591.6
	125	581.3	619.4	579.2	724.7	687.2	641.6	724.7
	250	673.4	735.4	697.5	831.4	806.6	741.6	831.4

Empirical studies

- Data on the SPDR S&P 500 ETF (SPY)
- ETF (exchange traded fund) which tracks the S&P 500 Index
- Daily, 15/06/04 - 13/06/14 (2497 obs)
- VaR computed w/ two models

Estimation

- With mean

$$
\begin{aligned}
y_t &= r + \lambda h_{t-1} + \sqrt{h_{t-1}} u_t \\
u_t &\sim i.i.d.\ N(0,1) \\
h_t &= \omega + \beta h_{t-1} + \alpha (u_t - \gamma \sqrt{h_{t-1}})^2
\end{aligned}
$$

- Without mean

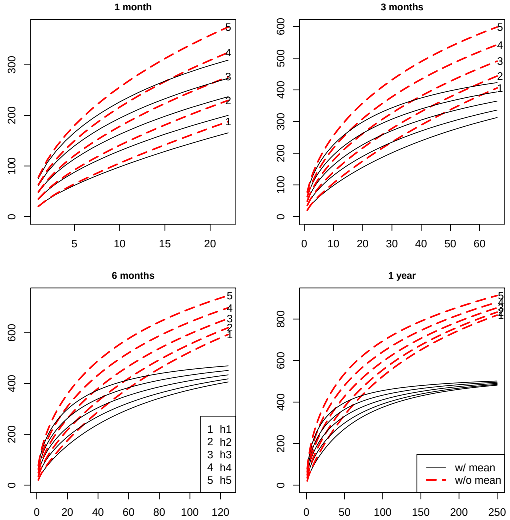
$$
\begin{aligned}
y_t &= \tilde{r} + \sqrt{h_{t-1}} u_t \\
u_t &\sim i.i.d.\ N(0,1) \\
h_t &= \omega + \beta h_{t-1} + \alpha (u_t - \gamma \sqrt{h_{t-1}})^2
\end{aligned}
$$

Estimation result

	w/ mean		w/o mean
	Estimate	Std error	Estimate
λ	4.22	5.122	0
r	5.31e-10	4.63e-04	2.66e-07
ω	-1.04e-07	9.00e-07	-4.22e-08
α	2.96e-06	6.59e-07	3.07e-06
β	7.96e-01	5.98e-02	7.94e-01
γ	2.43e+02	6.22e+01	243e+02
Log likelihood	8079.7659		8076.6208

- Confidence interval of λ inverted Likelihood ratio test
- $[0.914, 7.526]$

Empirical results



- VaR with two different model yields difference
- Do not know which is better....

Empirical results

	Days	w/mean	w/o mean	diff
h_1	1	19.9	20.1	0.2
	10	98.2	105.7	7.5
	22	165.5	187.8	22.3
	66	312.7	406.1	93.4
	125	406.2	593.1	186.9
	250	481.5	818.8	337.4
h_2	1	34.2	34.8	0.6
	10	129.9	141.0	11.1
	22	200.1	229.8	29.6
	66	336.3	443.8	107.4
	125	418.4	619.8	201.4
	250	485.2	833.4	348.2

	Days	w/mean	w/o mean	diff
h_3	1	47.8	48.6	0.8
	10	162.5	178.3	15.7
	22	237.0	276.3	39.3
	66	364.6	490.7	126.0
	125	434.1	655.8	221.6
	250	490.0	854.4	364.4
h_4	1	61.3	62.8	1.5
	10	194.5	216.3	21.9
	22	273.1	324.3	51.2
	66	393.8	542.5	148.7
	125	451.5	698.2	246.6
	250	495.7	881.0	385.4
h_5	1	75.2	77.6	2.3
	10	226.7	255.8	29.1
	22	309.3	374.7	65.4
	66	422.9	598.5	175.5
	125	469.6	746.5	276.9
	250	501.4	913.3	412.0

Conclusion

Results so far w/ two scenarios

- VaR computed w/ "misspecified" model deviates from the oracle VaR
- H_0 : constant mean tends to be "accepted"
- VaR computed with estimated par. with μ_t better in expectation than μ
- Empirical studies: models w/ and w/o mean yields different VaR

From now

- What to do when DGP is unknown?
- Combine forecast?
- Robust control analysis (Hansen-Sargent 01)?

Pastor-Stambaugh(12)

Predictive system

$$\begin{aligned} y_t &= \mu_{t-1} + u_t \\ \mu_t &= c + \theta\mu_{t-1} + w_t \\ x_t &= c_x + Ax_{t-1} + v_t \end{aligned}$$

where $x_t \in \mathrm{R}^n$.

$$\begin{pmatrix} u_t \\ v_t \\ w_t \end{pmatrix} \sim N\left(0, \begin{pmatrix} \sigma_u^2, & \sigma_{uv}, & \sigma_{uw} \\ & \sigma_v^2 & \sigma_{vw} \\ & & \sigma_w^2 \end{pmatrix}\right)$$

x_t

- Dividend yield on U.S. equity
- First diff. in the long-term high-grade bond yield
- Diff. of long-term bond yield and short-term interest rate

Why these parameters?

- PS(12) conduct Bayesian estimation for the same model using annual returns
- Set par. values (for daily return) to match the aggregated annual returns

Proposition

Let $\{y_{tk}^{(k)}, t = 1, 2, \cdots\}$ be k-day aggregated returns, i.e.,

$$y_{tk}^{(k)} = y_{tk-(k-1)} + y_{tk-(k-2)} + \cdots + y_{tk-1} + y_{tk}.$$

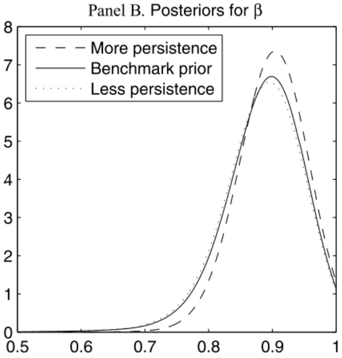
There exist $\{\mu_{tk}^{(k)}, t = 1, 2, \cdots\}$ such that

$$\begin{aligned} y_{tk}^{(k)} &= \mu_{tk}^{(k)} + u_{tk}^{(k)} \\ \mu_{tk}^{(k)} &= c^{(k)} + \theta^k \mu_{(t-1)k}^{(k)} + w_{tk}^{(k)} \end{aligned}$$

as a function of μ_t, u_t, w_t, and parameters in 1-day model.

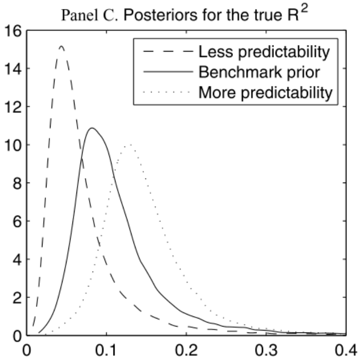
Back

Why these parameters? θ



- $\theta = 0.999$ (Benchmark) $\Rightarrow$ Annual $\theta = 0.999^{250} \approx 0.78$
- $\theta = 0.99 \Rightarrow$ Annual $\theta = 0.99^{250} \approx 0.08$
- $\theta = 0.98 \Rightarrow$ Annual $\theta = 0.98^{250} \approx 0.006$
- $\theta = 0.95 \Rightarrow$ Annual $\theta = 0.95^{250} \approx 2.7e - 6$

Why these parameters? σ_μ^2/σ_y^2

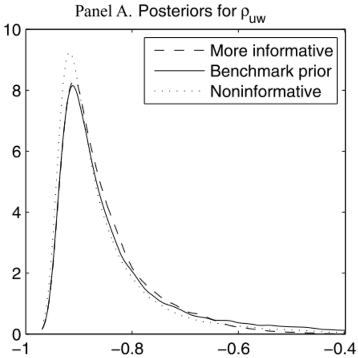


- $\sigma_\mu^2/\sigma_y^2 = 0.0003(\text{Benchmark}) \Rightarrow$ Annual counterpart ≈ 0.06
- $\sigma_\mu^2/\sigma_y^2 = 0.00005 \Rightarrow$ Annual counterpart ≈ 0.01
- $\sigma_\mu^2/\sigma_y^2 = 0.0005 \Rightarrow$ Annual counterpart ≈ 0.10
- $\sigma_\mu^2/\sigma_y^2 = 0.001 \Rightarrow$ Annual counterpart ≈ 0.20

Annual counterpart of σ_μ^2/σ_y^2

		σ_μ^2/σ_y^2			
		0.00005	0.0003	0.0005	0.001
	0.999	0.010	0.063	0.105	0.201
θ	0.99		0.012		
	0.98		0.0036		
	0.95		0.00056		

Why these parameters? ρ



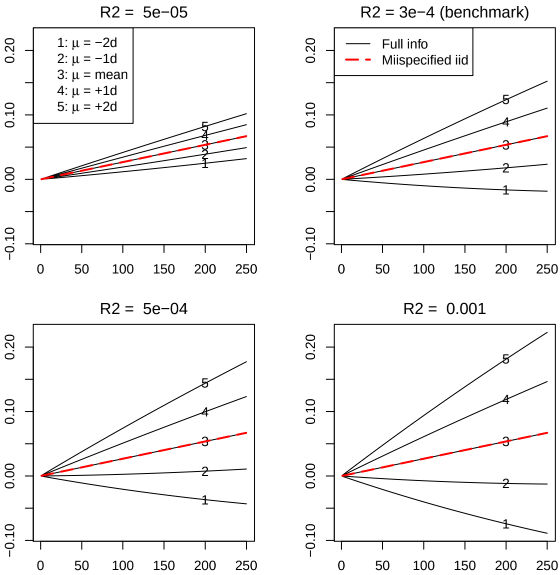
$\rho = -0.8$

- Benchmark θ & $\sigma_\mu^2/\sigma_y^2 \Rightarrow$ Annual $\rho = -0.77$

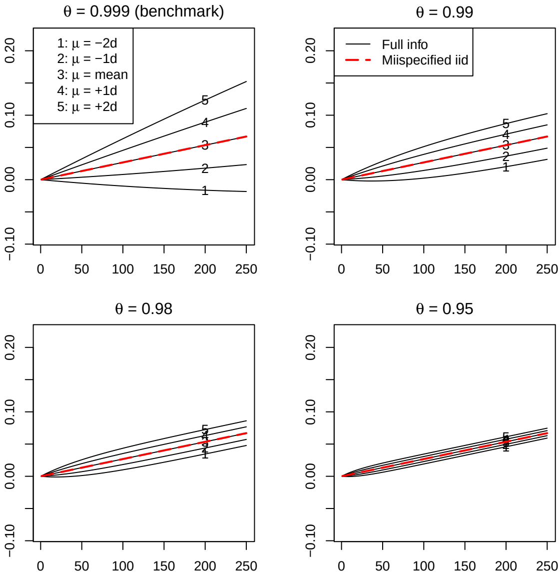
Annual counterpart of ρ

		σ_μ^2/σ_y^2			
		0.00005	0.0003	0.0005	0.001
	0.999	-0.785	-0.765	-0.753	-0.729
θ	0.99		-0.604		
	0.98		-0.461		
	0.95		-0.266		

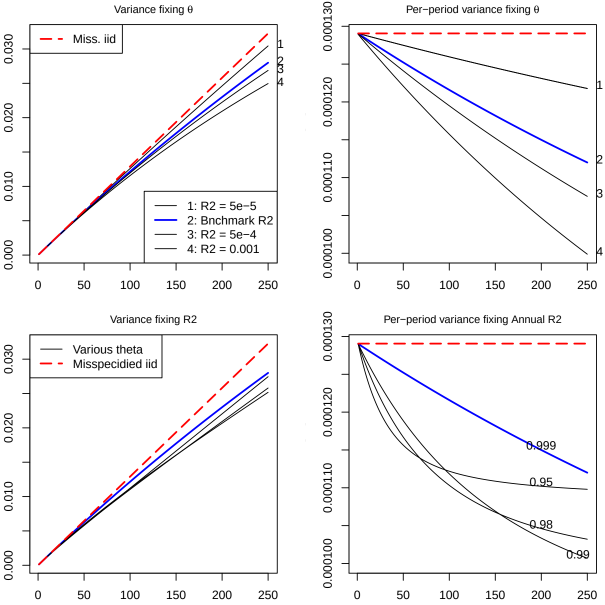
Mean: with various θ



Mean: with various σ_μ^2/σ_y^2

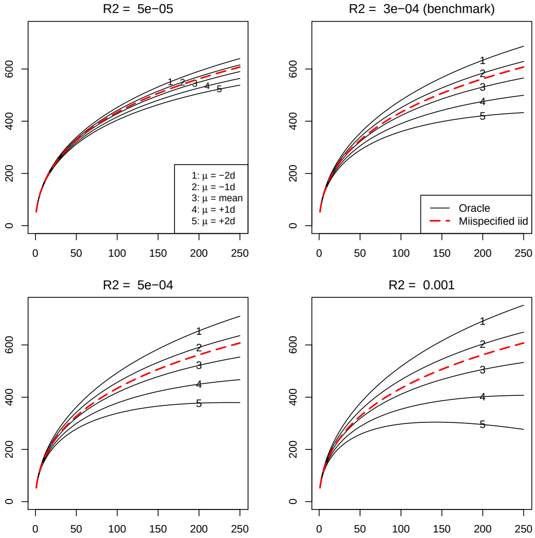


Variance



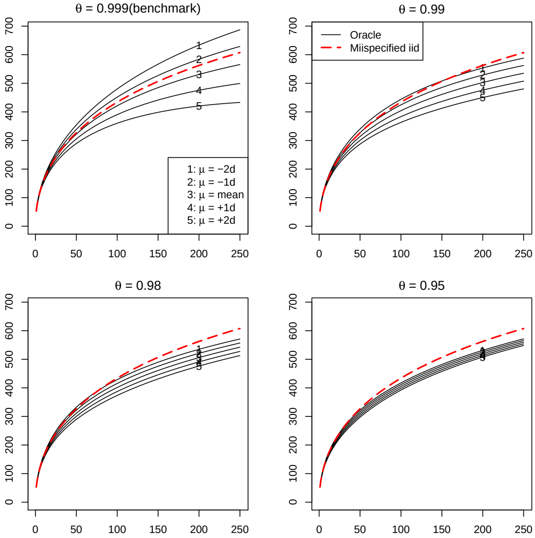
Variance fixing θ
Miss. iid
1: R2 = 5e−5
2: Bnchmark R2
3: R2 = 5e−4
4: R2 = 0.001
1
2
3
4
0.000
0.010
0.020
0.030
0
50
100
150
200
250
Per−period variance fixing θ
0.000100
0.000110
0.000120
0.000130
1
2
3
4
Variance fixing R2
Various theta
Misspecidied iid
Per−period variance fixing Annual R2
0.999
0.95
0.98
0.99

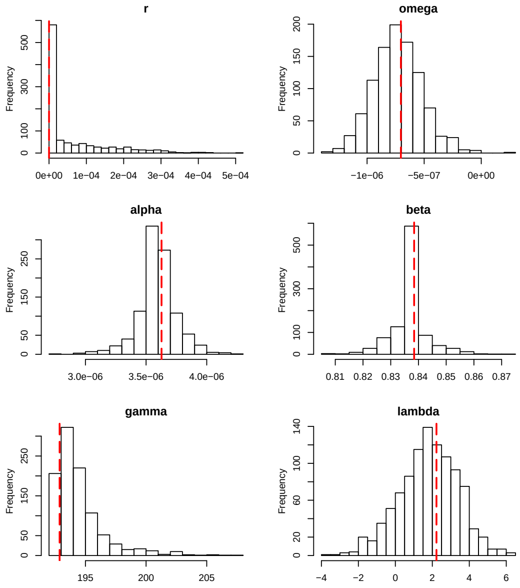
VaR: various θ

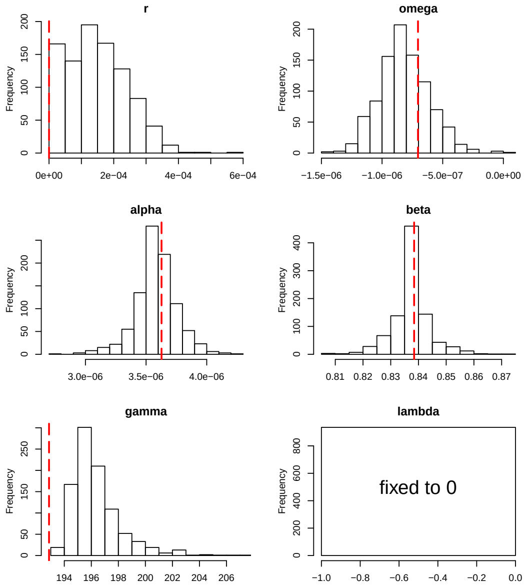


Back

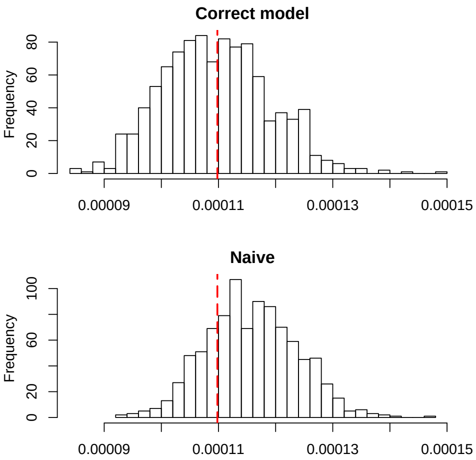
VaR: various σ_μ^2/σ_y^2



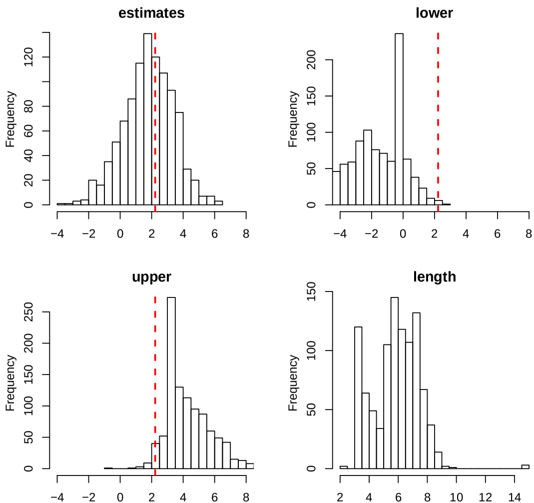

r
Frequency
0
100
300
500
0e+00
1e−04
2e−04
3e−04
4e−04
5e−04
omega
Frequency
0
50
100
150
200
−1e−06
−5e−07
0e+00
alpha
Frequency
0
50
150
250
3.0e−06
3.5e−06
4.0e−06
beta
Frequency
0
100
300
500
0.81
0.82
0.83
0.84
0.85
0.86
0.87
gamma
Frequency
0
50
150
250
195
200
205
lambda
Frequency
0
20
60
100
140
−4
−2
0
2
4
6



r
Frequency
0
50
100
150
200
0e+00
2e−04
4e−04
6e−04
omega
Frequency
0
50
100
150
200
−1.5e−06
−1.0e−06
−5.0e−07
0.0e+00
alpha
Frequency
0
50
100
200
3.0e−06
3.5e−06
4.0e−06
beta
Frequency
0
100
200
300
400
0.81
0.82
0.83
0.84
0.85
0.86
0.87
gamma
Frequency
0
50
150
250
194
196
198
200
202
204
206
lambda
Frequency
0
200
400
600
800
fixed to 0
−1.0
−0.8
−0.6
−0.4
−0.2
0.0



Correct model
Frequency
0
20
40
60
80
0.00009
0.00011
0.00013
0.00015
Naive
Frequency
0
20
60
100
0.00009
0.00011
0.00013
0.00015



estimates
lower
upper
length
Frequency
Frequency
Frequency
Frequency

Why $corr(u_t, w_t)$ likely to be negative?

- By loglinear decomposition by Campbell 91,

$$
\begin{aligned}
u_{t+1} &\approx (E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j \Delta d_{t+1+j} - (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j y_{t+1+j} \\
&\equiv u_{d,t+1} - u_{r,t+1}
\end{aligned}
$$

- Given the form of y_t, $u_{r,t+1} = \kappa w_{t+1}$ with $\kappa > 0$.

$$
\begin{aligned}
cov(u_t, w_t) &= cov(u_{d,t+1} - \kappa w_{t+1}, w_{t+1}) \\
&= cov(u_{d,t+1}, w_{t+1}) - \kappa var(w_{t+1})
\end{aligned}
$$

- If cashflow is deterministic, $cov(u_t, w_t) < 0$ immediately.
- Stock : cashflow (d_{t+j}) much less volatile than discount rate (y_{t+j}) (Excess volatility puzzle (Shiller 81))
- The second term seems to dominate first term

Back

Why $V_{correct} < V_{miss}$ if σ_μ^2/σ_y^2 is low?

For example, when $\tau = 2$,

$$\begin{aligned}
V_{correct,tau=2} & \\
= Var_t(y_{t+1} + y_{t+2}|\mu_t) &= Var_t(\mu_t + u_{t+1} + \underbrace{\mu_{t+1}}_{=c+\theta\mu_t+w_{t+1}} + u_{t+2}) \\
&= Var_t(u_{t+1} + u_{t+2} + w_{t+1}) \\
&= 2\sigma_u^2 + 2\sigma_{uw} + \sigma_w^2 \\
&= 2\sigma_u^2 + \underbrace{2\rho\sigma_u\sigma_w}_{<0} + \sigma_w^2 \\
V_{miss,\tau=2} &= 2\sigma_u^2 + 2\sigma_\mu^2
\end{aligned}$$

- $\sigma_u >> \sigma_\mu > \sigma_w$
  - $\sigma_\mu^2 = 0.0003\sigma_y^2 = \frac{0.0003}{0.9997}\sigma_u^2 \approx 0.0003\sigma_u^2$
  - $\sigma_w^2 = (1-\theta^2)\sigma_\mu^2 \approx 0.0020\sigma_\mu^2$
- $\rightarrow \sigma_u\sigma_w > \sigma_\mu^2$