High Dimensional Multivariate Realized Volatility Measures

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The general setup and motivation

- p assets, p large;
- Frictionless log-prices: a p-dimensional *Itô* process with factors

$$dP_t^* = b \cdot \sigma_{ft} dB_t^F + \sigma_{\epsilon t} dB_t^I \quad (1)$$

- Object of interest: the integrated covolatility matrix of asset prices

$$\Sigma_T = b \cdot \int_0^T \left(\sigma_{ft} \sigma'_{ft} \right) dt \cdot b' + \int_0^T \left(\sigma_{\epsilon t} \sigma'_{\epsilon t} \right) dt \quad (2)$$

- Σ_T : Important for asset pricing, portfolio allocation and risk management;
  - Optimal portfolio allocation: function of Σ_T^{-1};
  - Risk of a portfolio: related to Σ_T.
- The portfolio allocation or the risk management can involve a large number of assets;
- Need to have an accurate and well conditioned estimator for a large number of assets.

Related literature

- Estimators of the covolatility for a small number of assets: Hayashi et al. (2005); Barndorff-Nielsen et al.(2008); Christensen et al. (2010);
- High dimensional cases: Nerlov and Diebold (1989); Shephard et al. (2006); Fan et al. (2008); Wang et al.(2010); Lunde et al. (2011); Tao et al. (2011) and Bannouh et al. (2012); Ait-Sahalia et al. (2016)
- For most of these papers: sparsity; factor structure of returns; observable factors;
- Problem: no consensus on factors to use; misspecification and missing factors problems;
- **This paper**: factor structure of returns; unobserved factors; **microstructure noise**; **factor structure on microstructure noise**;
- *Advantage of a factor model*: a semi-definite positive estimator; Estimator is invertible under weak conditions; Reduction of the number of parameters to estimate.

Key findings

- When the number of assets is large, a factor structure in returns and in microstructure noise improve the estimation of:
  - The integrated covolatility matrix;
  - The correlation matrix;
  - The inverse of the integrated covolatility matrix.

Outline

The setup

- Δ: frequency of the data
- $dP_t^* = b \cdot dF_t + dE_t \Longrightarrow \int_{t-\Delta}^{t} dP_s^* = b \cdot \int_{t-\Delta}^{t} dF_s + \int_{t-\Delta}^{t} dE_s$
- Notations:
  - $r_t^* \equiv r_{t,\Delta}^* \equiv \int_{t-\Delta}^{t} dP_s^*$;
  - $f_t \equiv f_{t,\Delta} \equiv \int_{t-\Delta}^{t} dF_s$;
  - $\epsilon_t \equiv \epsilon_{t,\Delta} \equiv \int_{t-\Delta}^{t} dE_s$
- Matricial representation:

$$r_t^* = bf_t + \epsilon_t \tag{3}$$

- $f_{kt} \perp f_{lt}$, $\forall k \neq l$;
- $f_{kt} \perp \epsilon_{it}$;
- $\epsilon_{it} \perp \epsilon_{jt}$, $\forall i \neq j$;
- $E(\epsilon_{it} | I_{t-\Delta}) = 0$

The setup

- The covolatility matrix
$$\Sigma = b\Sigma_f b' + \Sigma_\epsilon \tag{4}$$
- Parameters of interest: Σ_f, Σ_ϵ, and b
- Observed log-prices are noisy
$$P_t = P_t^* + u_t \tag{5}$$
- Factor representation of the microstructure noise:
$$u_t = cg_t + \eta_t \tag{6}$$
- $\eta_t \perp g_t, f_t, \epsilon_t, P_t^*$;
- $g_t \perp f_t, \epsilon_t, P_t^*$;
- g_t and η_t: independent across time;
- c is a $p \times K'$ vector of noise loadings;
- g_t is $K' \times 1$: captures the cross-sectional dependence in the measurement errors

Evidence of a factor structure in microstructure noise

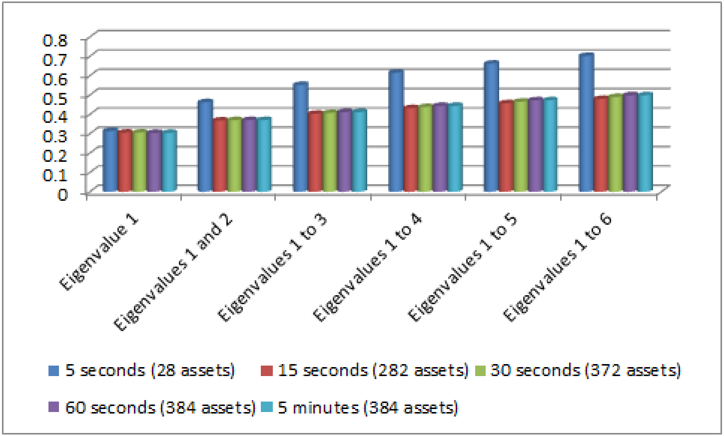
- 384 stocks traded on $S\&P500$;
- Six random trading days;
- For each day d, estimation of the covolatility of the microstructure noise:

$$\hat{\Sigma}_u^d = \frac{\sum_{i=1}^n r_{j/n} r'_{j/n}}{2n}$$

- Run the spectral decomposition of $\hat{\Sigma}_u^d$, $d = 1, ..., 6$ to get eigenvalues;
- For different sampling frequency, compute the share of the total variability of the microstructure noise explained 1, 2, ..., 6 factors.

Evidence of a factor structure in microstructure noise

Figure: Ratio of largest eigenvalues relative to the total variation

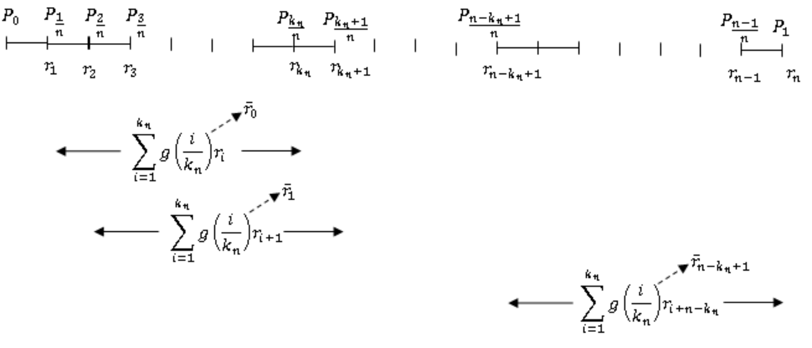


Estimation: recall on the Pre-averaging approach

- Observed returns are also noisy

$$r_{it} = r^*_{it} + (u_{it} - u_{it-\Delta})$$

Figure: The pre-averaging approach



- The probability order of the microstructure noise part moves from $O_p(1)$ to $O_p(\frac{1}{\sqrt{k_n}})$.

Estimators based on the Pre-averaging approach

- Pre-averaging estimator of the integrated volatility (Jacod et al. (2009))

$$PRV(r) = \frac{1}{\theta\psi_2\sqrt{n}} \sum_{j=0}^{n-k_n+1} (\bar{r}_j)^2 - \frac{\psi_1}{2\theta^2\psi_2 n} \sum_{i=1}^{n} (r_i)^2 \qquad (7)$$

- Pre-averaging estimator of the integrated covolatility (Christensen et al. (2010))

$$MRC\,[r]_n = \frac{n}{(n-k_n+2)} \frac{1}{\psi_2 k_n} \sum_{i=0}^{n-k_n+1} \bar{r}_i \bar{r}_i' - \frac{\psi_1^{k_n}}{2n\theta^2\psi_2^{k_n}} \sum_{i=1}^{n} r_i r_i' \qquad (8)$$

θ, ψ_1 and ψ_2: setting parameters; k_n: number of averaged return per block; $\bar{r}_i$ the i^{th} averaged return, r_i the i^{th} usual return.

The different steps: an overview

- Estimation of rotations of the factor components using the factor analysis;
- Estimation of the integrated volatility of factors;
- Estimation of factor loadings;
- Estimation of idiosyncratic components;
- Estimation of the integrated volatility of idiosyncratic components;
- Deduction of $\hat{\Sigma}$ by a natural plug-in.

Estimation of a rotation of factors

- Δ: The observation frequency assumed to be constant over time and over assets, and closed to zero;
- Estimation of the rotated factor $\tilde{f}$: minimization of the scaled sum of squared values of the idiosyncratic component:

$$\begin{cases} \underset{f_{j\Delta},b}{Min} \;\; \frac{1}{p} \sum_{j=1}^{\lfloor 1/\Delta \rfloor} (r^*_{j\Delta} - bf_{j\Delta})'(r^*_{j\Delta} - bf_{j\Delta}) \\ s.t \;\; \frac{1}{p} b'b = I_K \end{cases} \tag{9}$$

- The solution of the minimization problem

$$\hat{\tilde{f}}_{j\Delta} = \frac{1}{p} W' r^*_{j\Delta}, \quad \forall j = 1, ..., \lfloor 1/\Delta \rfloor$$

- W is the matrix of ordered eigenvectors of $\sum_{j=1}^{\lfloor 1/\Delta \rfloor} r^*_{j\Delta} r^{*'}_{j\Delta}$.

Estimation of a rotation of factors

- The continuous time solution: by taking $\Delta \longrightarrow 0$

$$\hat{\hat{f}}_t = \frac{1}{p} W' r_t^*, \quad \forall t > 0 \tag{10}$$

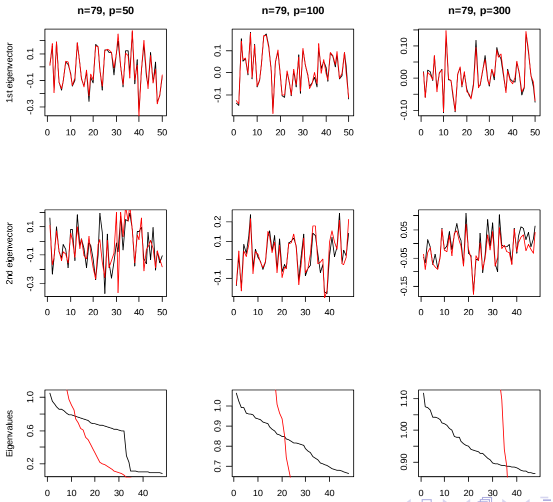
- Columns of W correspond to ordered eigenvectors of Σ;
- W and r^* are latent: unfeasible estimator;
- As feasible estimator:

$$\hat{f}_t = \frac{1}{p} \hat{W}' r_t, \quad \forall t > 0 \tag{11}$$

- $\hat{W}$: the matrix of K ordered eigenvectors of MRC (or another consistent estimator in the small dimensional framework as the Kernel estimator $MRker$ defined by $MRker(r) = \sum_{h=-n}^{n} k(\frac{h}{H+1}) \sum_{j=h+1}^{n} r_j r'_{j-h}$),
- r_t the series of the $p \times 1$ vector of observed returns.

Why to use eigenvectors of Kernel or Pre-averaging covolatility estimators?

Figure: Spectral decomposition of $MRker$



Estimation of parameters of interest

- $\hat{f}_t$ can be decomposed as

$$
\begin{aligned}
\hat{f}_{kt} &= \frac{1}{p}W_k'r_t^* + \frac{1}{p}W_k'(u_t - u_{t-\Delta}) + \frac{1}{p}W_k^{\epsilon'}r_t^* + \frac{1}{p}W_k^{\epsilon'}(u_t - u_{t-\Delta}) \\
&\approx \frac{1}{p}W_k'bf_t + \frac{1}{p}W_k'\epsilon_t + \frac{1}{p}W_k'(u_t - u_{t-\Delta}) \\
&\approx \frac{1}{p}W_k'bf_t + \frac{1}{p}W_k'(u_t - u_{t-\Delta}) \\
&\approx \tilde{f}_{kt} + \frac{1}{p}W_k'(u_t - u_{t-\Delta}) \\
&\approx \tilde{f}_{kt} + \frac{1}{p}W_k'c(g_t - g_{t-\Delta}) + O_p(p^{-1/2}n^{-1/2})
\end{aligned}
$$

- $\hat{f}$: consistently estimate a rotation of the latent factor f contaminated by microstructure noises.

Estimation of parameters of interest

- **Remark:** Let H be a $K \times K$ orthogonal matrix ($H'H = I_K$). Then

$$\begin{aligned} r_t^* &= bf_t + \epsilon_t \\ &= bI_K f_t + \epsilon_t \\ &= bH'Hf_t + \epsilon_t \\ &= \tilde{b}\tilde{f}_t + \epsilon_t, \quad \textit{with} \quad \tilde{b} = bH' \textit{ and } \tilde{f}_t = Hf_t \end{aligned}$$

- *Conclusion:* Σ is unchanged by the same rotation of the factors and loadings;

Estimator of the integrated volatility of $\tilde{f}$

- $\hat{f}_{kt} = \tilde{f}_{kt} + \frac{1}{p}W_k' c(g_t - g_{t-\Delta}) + O_p(p^{-1/2}n^{-1/2})$
- $\int_0^1 \widehat{\sigma^2_{\tilde{f}ku}} du = PRV(\hat{f}_k)$

- PRV() is the pre-averaging estimator defined by (8).

Estimation of parameters of interest

- *Notation:* $[X, Y]$ denotes the integrated covolatility between X and Y;

$$\begin{aligned}\left[r_i^*, \tilde{f}_k\right] &= \left[\tilde{b}_{ik}\tilde{f}_k + \epsilon_i, \tilde{f}_k\right] \\ &= \tilde{b}_{ik}\left[\tilde{f}_k, \tilde{f}_k\right] + \left[\epsilon_i, \tilde{f}_k\right] \\ &= \tilde{b}_{ik}\int_0^1 \sigma^2_{\tilde{f}_{kt}}\, dt\end{aligned}$$

Estimator of the loadings

- $\left[r_i^*, \tilde{f}_k\right] = \tilde{b}_{ik}\int_0^1 \sigma^2_{\tilde{f}_{kt}}\, dt \Longrightarrow \hat{b}_{ik} = \frac{MRC\left(r_i, \hat{f}_k\right)}{PRV\left(\hat{f}_k\right)}$

- MRC is the Modulated realized covariance defined by (10).

Estimation of parameters of interest

For n and p are sufficiently large

$$\begin{aligned} \hat{\epsilon}_{it} &= r_{it} - \sum_{k=1}^{K} \hat{b}_{ik} \hat{f}_{kt} \\ &\approx r_{it} - \sum_{k=1}^{K} \tilde{b}_{ik} \tilde{f}_{kt} - \frac{1}{p} \sum_{k=1}^{K} \tilde{b}_{ik} W_k' c(g_t - g_{t-\Delta}) \\ &\approx \epsilon_{it} + (u_t - u_{t-\Delta}) - \frac{1}{p} \sum_{k=1}^{K} \tilde{b}_{ik} W_k' c(g_t - g_{t-\Delta}) \end{aligned}$$

$\hat{\epsilon}_{it}$ consistently estimates ϵ_{it} plus a microstructure noise term;

Estimator of the integrated volatility of the idiosyncratic components

- $\widehat{\int_0^1 \sigma_{\varepsilon iu}^2} du = PRV(\hat{\epsilon}_i)$

Estimator of the covolatility matrix

- $\widehat{\Sigma}$ is obtained by plug in the estimators of $\tilde{b}$, $\int_0^1 \sigma^2_{fku} du$ for $k = 1, ..., K$ and $\int_0^1 \sigma^2_{\varepsilon iu} du$, for $i = 1, ..., p$ in (16)

The covolatility matrix estimator

$\forall i, j = 1, ..., p$

- $\widehat{\Sigma}_{ij} = \sum_{k=1}^{K} \frac{MRC(r_i, \hat{f}_k).MRC(r_j, \hat{f}_k)}{PRV(\hat{f}_k)}$
- $\widehat{\Sigma}_{ii} = \sum_{k=1}^{K} \frac{MRC(r_i, \hat{f}_k)^2}{PRV(\hat{f}_k)} + PRV(\hat{\epsilon}_i)$

- PRV and MRC are respectively the uni dimensional and the multidimensional pre-averaging estimators.

Competitors

- The Multidimensional Realized kernel

$$MRker(r) = \sum_{h=-n}^{n} k(\frac{h}{H+1})\Gamma_h, \tag{12}$$

$$\Gamma_h = \sum_{j=h+1}^{n} r_j r'_{j-h}, \text{ for } h > 0; \ \Gamma_h = \Gamma'_{-h}, \text{ for } h < 0.$$

- The adjusted Modulated Realized Covariance

$$MRC\,[r]_n^{\delta} = \frac{n}{(n-k_n+2)} \frac{1}{\psi_2 k_n} \sum_{i=0}^{k_n} (\bar{r}_i)\,(\bar{r}_i)' \tag{13}$$

- The composite Kernel $\hat{\Sigma}_{comp}$: uni-realized Kernel for diagonal elements, bivariate realized kernel for off-diagonal terms and regularization.

Rates of convergence

Lemma

Under assumptions of the model and the Frobenius norm, we have the following results

- $\left|\hat{\Sigma}^{\tilde{f}}_{kk} - \Sigma^{\tilde{f}}_{kk}\right| = O_p\left(n^{-1/4}\right)$, $\forall\ k = 1, ..., K$
- $\left\|\hat{b}_k - b_k\right\|_F = O_p\left(p^{1/2} n^{-1/4}\right)$, $\forall\ k = 1, ..., K$
- $\left\|\hat{\Sigma}^{\epsilon} - \Sigma^{\epsilon}\right\|_F = O_p(p^{1/2} n^{-1/4})$

Theorem

Under assumptions of the model, we have

- $\left\|\hat{\Sigma} - \Sigma\right\|_F = O_p(pn^{-1/4})$
- $\left\|MRC^{\delta} - \Sigma\right\|_F = O_p(pn^{-1/5})$
- $\|MRker - \Sigma\|_F = O_p(pn^{-1/5})$
- $\left\|\hat{\Sigma}_{comp} - \Sigma\right\|_F = O_p(\sqrt{p(p-1)}n^{-1/5})$

Simulation design

- The factor component:

$$f_{kt} = \sigma_{f_{kt}} dB_t^F, \; k = 1, 2$$

B_t^F a brownian motion and $\sigma_{ft} \sim GARCH$ diffusion model as in Torben G. Andersen et al. (2006);

- The idiosyncratic error term:

$$\varepsilon_{it} = \sigma_{\epsilon it} dB_{it}^I$$

  - For $1 \leq i \leq p/3$, $\sigma_{\epsilon it} \sim$ *Nelson GARCH* diffusion limit model as in Bandorff-Nielsen and Shephard (2002);
  - For $p/3 < i \leq 2p/3$, $\sigma_{\epsilon it} \sim OU$ model as in Bandorff-Nielsen and Shephard (2002);
  - For $2p/3 < i \leq p$, $\sigma_{\epsilon it} \sim GARCH$ diffusion model as in Torben G. Andersen et al. (2006).

- The microstructure noise: one factor.

Simulation results: the synchronous case

Table: Scaled Frobenius norm $\|A\|_F = \sqrt{\frac{1}{p}\left(\sum_{i=1}^{p}\sum_{j=1}^{p}\left|a_{ij}^2\right|\right)}$ of the estimation errors for the factorial estimator, the composite kernel, the multivariate Kernel and the adjusted modulated realized covariance: data are synchronous and low noise ($\xi^2 = 0.001$); $n = 78$

	Covariance	Correlation	Inverse	PSD	Diag	Off-Diag
Number of assets: p=50						
$\hat{\Sigma}$	**1.206**	**0.921**	**2.271**	**1**	**5.183**	**75.007**
	(0.283)	*(0.190)*	*(5.389)*			
$\hat{\Sigma}_{comp}$	1.541	1.055	3.170	1	11.940	119.563
	(0.292)	*(0.131)*	*(0.197)*			
MRker	1.587	1.128	270.884	1	6.281	134.326
	(0.279)	*(0.129)*	*(74.929)*			
MRC^{δ}	1.560	1.116	147.141	1	6.025	128.28
	(0.279)	*(0.132)*	*(395.209)*			
Number of assets: p=100						
$\hat{\Sigma}$	**2.310**	**1.221**	**2.895**	**1**	**18.498**	**538.322**
	(0.485)	*(0.205)*	*(34.790)*			
$\hat{\Sigma}_{comp}$	2.730	1.418	3.835	1	62.401	910.124
	(0.944)	*(0.186)*	*(0.213)*			
MRker	2.778	1.498	NA	0	27.7020	963.417
	(0.923)	*(0.184)*	NA			
MRC^{δ}	2.698	1.480	NA	0	24.707	891.115
	(0.864)	*(0.190)*	*NA*			

Simulation results: the asynchronous case

Table: Scaled Frobenius norm $\|A\|_F = \sqrt{\frac{1}{p}\left(\sum_{i=1}^{p}\sum_{j=1}^{p}\left|a_{ij}^2\right|\right)}$ of the estimation errors for the factorial estimator, the composite kernel, the multivariate Kernel and the adjusted modulated realized covariance: data are asynchronous and low noise ($\xi^2 = 0.001$)

	Covariance	Correlation	Inverse	PSD	Diag	Off-Diag
Number of assets: p=50						
$\hat{\Sigma}$	**4.559**	**1.648**	**3.424**	**1**	**94.875**	**963.622**
	(0.317)	*(0.120)*	*(0.196)*			
$\hat{\Sigma}_{comp}$	4.790	1.629	3.250	1	90.197	1075.323
	(0.279)	*(0.069)*	*(0.141)*			
MRker	4.906	1.710	229.652	1	89.754	1141.481
	(0.279)	*(0.079)*	*(111.995)*			
MRC^{δ}	4.829	1.700	128.104	1	87.405	1110.009
	(0.278)	*(0.079)*	*(62.922)*			
Number of assets: p=100						
$\hat{\Sigma}$	**6.329**	**2.131**	**3.426**	**1**	**170.842**	**3912.406**
	(0.448)	*(0.141)*	*(0.595)*			
$\hat{\Sigma}_{comp}$	6.908	2.347	3.558	1	146.474	4678.125
	(0.400)	*(0.083)*	*(0.133)*			
MRker	7.233	2.511	NA	0	165.132	5061.262
	(0.412)	*(0.107)*	NA			
MRC^{δ}	7.002	2.416	NA	0	152.773	4752.494
	(0.422)	*(0.109)*	*NA*			

Empirical studies

- **Data**: Five years of intraday data, collected from the TAQ database of the Wharton Research Data Services (January 2007-December 2011);
- The choice of the number of factors as in Bai et al. (2002): minimizing the criteria

$$IC = log\left[\frac{1}{p}\sum_{j=1}^{\lfloor 1/\Delta \rfloor}(r_{j\Delta} - bf_{j\Delta})'(r_{j\Delta} - bf_{j\Delta})\right] + Kg(p, \lfloor 1/\Delta \rfloor) \qquad (14)$$

- Performance evaluation
  - Resolve the following classic Markowitz portfolio problem with a gross exposure constraint as in Fan et al. (2008)

$$\begin{cases} Min \quad w_{t+1}^{'}\hat{\Sigma}_t w_{t+1} \\ s.t\ w_{t+1}^{'}1 = 1 \quad and \quad \sum_{i=1}^{p}|w_{i,t+1}| \leq 1 + 2s \end{cases} \qquad (15)$$

  - The performance criteria

$$\hat{w}_{t+1}^{'} RCov_{t+1}\hat{w}_{t+1} \qquad (16)$$

where $RCov_{t+1}$ is the out of sample *5-minutes* realized covariance matri...

Empirical studies

Table: Performance evaluation of estimators of the covolatilaty matrix: Average out of sample risk under gross exposure constraints; liquid assets (traded at least 195 times)

	s=0	s=0.25	s=0.5	s=1
$\hat{\Sigma}$	**0.495**	**0.245**	**0.240**	**0.241**
$\hat{\Sigma}_{comp}$	0.520	0.301	0.325	0.326
	(0.004)	*(0.003)*	*(0.001)*	*(0.001)*
$MRKer$	0.545	0.278	0.263	0.258
	(0.04)	*(0.09)*	*(0.080)*	*(0.007)*
MRC^{δ}	0.559	0.362	0.343	0.319
	(0.014)	*(0.014)*	*(0.011)*	*(0.003)*
$EqualWeight$	0.636	0.636	0.636	0.636
	(6E-05)	*(5E-09)*	*(3E-09)*	*(2E-09)*

Conclusion

- We use a statististical factor model to estimate a high dimensional covolatility matrix;
- We provide evidence of a factor structure the microstructure noise;
- A factor structure in returns and in microstructure noise improve the estimation of the integrated covolatility matrix, the correlation matrix, the inverse of the integrated covolatility matrix.;
- This result is proved analytically, confirmed through a simulation exercise and an empirical study.

THANK YOU